Limit Theorems and Laws of Large Numbers

- Knowledge of just the mean and variance of a rv can provide bounds on probabilities.
- Consider the sequence of non-negative rvs $X$ with common distribution $f_X(x)$ Let $\mu_X = E[X]$ and $\sigma^2_X = Var[X]$ (assumed finite).

- **Markov’s Inequality**
  
  For any $\epsilon > 0$,
  
  $$P[X \geq \epsilon] \leq \frac{\mu_X}{\epsilon}$$

- **Chebyshev’s Inequality**
  
  For any general random variable $X$ and any value $k > 0$
  
  $$P[|X - \mu_X| \geq k] \leq \frac{\sigma^2}{k^2}$$
• **Weak Law of Large Numbers**

Let $X_i$ be a sequence of iid rvs and $S_n = X_1 + X_2 + \ldots + X_n$

For every $\epsilon > 0$, as $n \to \infty$:

$$P \left[ \left| \frac{S_n}{n} - \mu_X \right| > \epsilon \right] \to 0$$

The probability that the average $S_n/n$ differs from $\mu_X$ by less than an arbitrary value $\epsilon$ tends to one.

• **Central Limit Theorem**

Under the same conditions as above:

$$P \left[ \frac{S_n - n\mu_X}{\sigma_X \sqrt{n}} < \beta \right] \to \phi(\beta)$$

Where $\phi(\beta) : N(0, 1)$ Standard Normal Distribution

• This is stronger than the weak law of large numbers since it gives the probabilities of the difference, through the Gaussian distribution.

• But the law of large numbers holds even when the variance may not exist and therefore it is more general.
• **Strong Law of Large Numbers**

• This law states that the average of a sequence of iid rvs converges to the mean of the distribution with probability one.

\[
\frac{S_n}{n} \to \mu_X \quad \text{as} \quad n \to \infty \\
\mathbb{P} \left[ \lim_{n \to \infty} \left( \frac{S_n}{n} = \mu_X \right) \right] = 1
\]