Scattering of a spherical wave by a rigid wedge

Ambika Bhatta (ambika_bhatta@student.uml.edu), Charles Thompson
(Charles_Thompson@uml.edu), and Kavitha Chandra (Kavitha_Chandra@uml.edu)

Center for Advanced Computation and Telecommunications University of Massachusetts Lowell, Lowell MA 01854

Max Denis (maxfdenis@hotmail.com)

Mayo Clinic Rochester Minnesota

(Dated: June 24, 2016)

Abstract

This paper investigates the scattering of spherical waves from a two dimensional re-entrant corner. The exact solution is obtained in frequency and time domain. The image based representation of the point source yields a simple expression of the exact solution. Numerical results to validate the approach is also presented. It is shown that the new expression can easily be evaluated for very high frequency and long range. Therefore, for asymptotic solution stationary phase approximation is not required.

PACS numbers: 43.38.Hz, 43.30.Bi

I. INTRODUCTION

The scattering behavior of wedges from low to very high frequency has a significant application from indoor and outdoor boundary value problems. The objective of this work is rigrous derivation of Green's function of a rigid wedge.

The modern study of diffraction owes its origins to the initial observations in the 17th century by Grimaldi and Huygens[1] and in the 19th century to the analytical work of Fresnel[2] and Kirchoff. Sommerfeld's was first to obtain the solution for the diffraction of a spherical wave from a half plane by mathematical construction associated with method of images[16]. He extended this approach[4] to evaluate diffraction from corners and wedges. Sommerfeld's solution resolved the shortcoming of the Fresnel-Kirchoff diffraction and provided an exact solution for this problem. The problem of scattering from corners and wedges at moderate frequencies has remained a subject of continuing interest for the next 100 years.

In 1915 Bromwich[5] developed a method to evaluate the diffracted field from wedges where interior angle is a rational multiple of π. In his work the series obtained as the superposition of the response of images is replaced by an integral in complex plane. The integral was extended to generalize the expression for any wedge angle. Garnir[6] in 1952 developed the first closed form solution for scattering of a point source waves from a rigid wedge in Laplace domain. In 1956 by Oberhettinger[8] also presented the time domain solution in terms of modified Bessel functions tranforming from Laplace domain. Biot[7] in 1957 and Tolstoy[?] generalized the transient solution in normal coordinates based on the solution obtained by Garnir.

It is interesting to note that the Bromich integral was the basis for the solution obtained by A. D Rawlins[10] which was subsequently developed as integral in real plane[11] by direct implementation of Fourier transform in space and time. In 1972 J.H Thompson[9] presented the transient solution obtained from Fourier domain represented as integrals of elementary functions . In 1980 M.J Buckingham's[12] solution of the scattered responsefor Neumann and Dirichlet boundary conditions was also in Fourier domain and then transformed to obtain the transient solution. Aforementioned work involved the simplication of the solution my implementing image based representation of the source.

The early discussion of the solution in does not provide the numerical evaluation of the scattered response. Also one does not necessarily require the approach of normal coordinates to evaluate the exact solution.

This paper presents a simple and concise approach to obtain the exact solution in frequency and time domain of aforementioned problem. The Helmholtz equation is modified by Fourier transform in space. This work presents the developement of the solution valid for all frequencies. It focuses on the significance of the effectively two contributing terms obtained from image based representation of the source. It is also shown that the impulse response can be evaluated directly by inverse Fourier transform in time.

II. PROBLEM STATEMENT

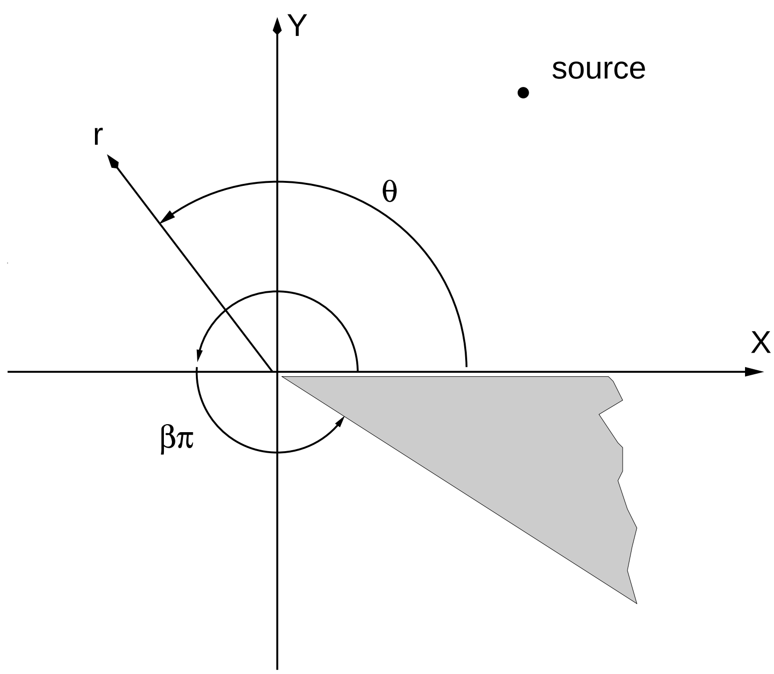

FIG. 1. Schematic of a rigid wedge

An infinite rigid wedge is shown in Fig.1. The apex of the wedge corresponds to the z-axis and the exterior angle is equal to $\beta\pi$ where $1 < \beta < 2$. The medium is excited by unit amplitude harmonically time varying point source with frequency ω located at $(r', \theta', 0)$. The responseis observed at (r', θ', z). In this case the complex amplitude of the response$G(r, \theta, z, \omega)$ will satisfy the inhomogeneous Helmholtz equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 G}{\partial \theta^2} + \frac{\partial^2 G}{\partial z^2} + k^2 G = \frac{\delta(\theta - \theta')\delta(r - r')\delta(z)}{r} \tag{1}$$

where $k = \frac{\omega}{c}$, c is the sound speed and the response satisfies the Neumann boundary condition at $\theta = 0$ and $\theta = \beta\pi$. The modal solution for the complex amplitude of the responseis obtained by first replacing $\delta(\theta - \theta')$ by an impluse train of like argument with angular period $2\pi\beta$.

$$G = -\frac{i}{\beta\pi}\sum_{m=0}^{\infty}\left(1 - \frac{\delta_m}{2}\right)\cos(\frac{m}{\beta}\theta)\cos(\frac{m}{\beta}\theta')I_{\frac{m}{\beta}}(r, r', z) \tag{2}$$

This choice allows one to express the response as a Fourier series in θ which yield the cosine terms that appear in the aforementioned expression. The coefficient of the series is

$$I_{\frac{m}{\beta}}(r, r', z) = \begin{cases} \int_0^\infty J_{\frac{m}{\beta}}(\hat{k}r)J_{\frac{m}{\beta}}(\hat{k}r')\left(\frac{e^{iz\sqrt{k^2-\hat{k}^2}}}{\sqrt{k^2-\hat{k}^2}}\right)\hat{k}d\hat{k} \\ -i\int_0^\infty J_{\frac{m}{\beta}}(\hat{k}r)J_{\frac{m}{\beta}}(\hat{k}r')\left(\int\limits_0^\infty J_0(\hat{k}\mu)\frac{e^{ik\sqrt{\mu^2+z^2}}}{\sqrt{\mu^2+z^2}}\mu d\mu\right)\hat{k}d\hat{k} \end{cases} \tag{3}$$

where the bracketed term is rewritten in terms of its Hankel transform[13]. To this end we first interchange the order of integration in Eqn.(3) and where split the integral in $\hat{k}$ into two integrals. The first integral contains the contribution over the interval $0 \leq \hat{k} \leq k$ and the second contribution $k \leq \hat{k} \leq \infty$.

$$I_{\frac{m}{\beta}}(r, r', z) = \frac{-i}{\pi}\int\limits_0^\infty\left(\int\limits_0^k J_{\frac{m}{\beta}}(\hat{k}r)J_{\frac{m}{\beta}}(\hat{k}r')J_0(\hat{k}\mu)\hat{k}d\hat{k} + \int\limits_k^\infty J_{\frac{m}{\beta}}(\hat{k}r)J_{\frac{m}{\beta}}(\hat{k}r')J_0(\hat{k}\mu)\hat{k}d\hat{k}\right)\frac{e^{ik\sqrt{\mu^2+z^2}}}{\sqrt{\mu^2+z^2}}\mu d\mu \tag{4}$$

The three-fold product of Bessel functions[?] applied. This process yields

$$I_{\frac{m}{\beta}}(r, r', z) = \frac{i}{\pi}\left(\frac{1}{rr'}\int_{|r'-r|}^{|r'+r|}\mu\frac{e^{ik\sqrt{\mu^2+z^2}}}{\sqrt{\mu^2+z^2}}\frac{\cos(\frac{m}{\beta}A)}{\sin A}\,d\mu \;-\; \frac{\sin(\frac{m}{\beta}\pi)}{rr'}\int_{|r'+r|}^{\infty}\mu\frac{e^{ik\sqrt{\mu^2+z^2}}}{\sqrt{\mu^2+z^2}}\frac{e^{-\frac{m}{\beta}B}}{\sinh B}\,d\mu\right) \tag{5}$$

where $\cos(A) = \frac{r^2+r'^2-\mu^2}{2rr'}$ for $0 \leq A \leq \pi$, $\cosh(B) = \frac{\mu^2-(r^2+r'^2)}{2rr'}$ for $B \geq 0$. Upon substitution of Eqn.(5) in to Eqn.(2) the infinite series including the integrals for long range and high frequencies requires more number of terms. This poses a numerical constraint on its evaluation. Therefore, the next section focuses on simplfying the expression for scattered response.

III. FREQUENCY DOMAIN SOLUTION

It will be shown that in all the cases of re-enterant corner the scattered response is a contribution of direct and the diffracted response. The change of variable of the integral as $\zeta = \cos^{-1}(\frac{r^2+r'^2-\mu^2}{2rr'})$ to the first integral in Eqn.(5) and for the second integral as $\zeta = \cosh^{-1}(\frac{\mu^2-(r^2+r'^2)}{2rr'})$ and substituting the result into Eqn.(2) yields the total scattered response as

$$G = G_{direct} + G_{diffracted} \tag{6}$$

where

$$G_{direct} = \frac{1}{\beta\pi^2}\int_0^{\pi} \frac{e^{ik\sqrt{r^2+r'^2-2rr'\cos\zeta+z^2}}}{\sqrt{r^2+r'^2-2rr'\cos\zeta+z^2}}\left[\sum_{m=0}^{\infty}(1-\frac{\delta_m}{2})\cos(\frac{m}{\beta}\theta)\cos(\frac{m}{\beta}\theta')\cos(\frac{m}{\beta}\zeta)\right]d\zeta \tag{7}$$

and

$$G_{diffracted} = -\frac{1}{\beta\pi^2}\int_0^{\infty} \frac{e^{ik\sqrt{r^2+r'^2+2rr'\cosh\zeta+z^2}}}{\sqrt{r^2+r'^2+2rr'\cosh\zeta+z^2}}\left(\sum_{m=1}^{\infty}\cos(\frac{m}{\beta}\theta)\cos(\frac{m}{\beta}\theta')\sin(\frac{m}{\beta}\pi)e^{-\frac{m}{\beta}\zeta}\right)d\zeta \tag{8}$$

The expression for the line-of-sight region will be evaluated first where G_{direct} in Eqn.(7) is the image based reflected and the direct response . The product of cosines enclosed in the square brackets will be rewritten as periodic sum of impulses with period $T = 2\pi\beta$ as Poisson's summation formula for source.

$$\sum_{m=0}^{\infty}(1-\frac{\delta_m}{2})\frac{\cos(\frac{m}{\beta}\theta)\cos(\frac{m}{\beta}\theta')\cos(\frac{m}{\beta}\zeta)}{\beta\pi} = \sum_{n=-\infty}^{\infty}\sum_{j=0}^{1}\sum_{m=0}^{1}\frac{\delta\left(\theta+(-1)^j\theta'+(-1)^m\zeta-nT\right)}{4} \tag{9}$$

Substituting Eqn.(9) into Eqn.(7) and noting the limits of the integration, we find that only impulses active in the interval $0 \leq \zeta \leq \pi$ need be retained. The locations of this these impulse are given by the values of ζ_1 through ζ_4 which are equal to $\theta-\theta'$, $\theta'-\theta$, $\theta+\theta'$, and $2\pi\beta-\theta-\theta'$ respectively and

$\zeta_5 = 0$; $\zeta_6 = 0$ for $\theta + \theta' = 0$ and $\theta' + \theta = 2\beta\pi$ respectively. ζ_5 and ζ_6 contribute when the source and the receiver are located on like edges of the wedge. Integrating Eqn.(7) yields

$$G_{direct} = \sum_{n=1}^{4} \frac{e^{ikR(\zeta_n)}}{4\pi R(\zeta_n)} \left[u(\zeta_n) - u(\zeta_n - \pi)\right] + \frac{e^{ikR(0)}}{4\pi R(0)} \left[\frac{\delta_{\theta+\theta'}}{2} + \frac{\delta_{2\pi\beta-\theta-\theta'}}{2}\right] \tag{10}$$

where $R(\zeta) = \sqrt{r^2 + r'^2 - 2rr' \cos\zeta + z^2}$ and $u(\zeta)$ is defined as the Heaviside step function The factor of $\frac{1}{2}$ in the square bracketed term in Eqn.(10) is the result of integration over half of the Kronecker delta function when ζ_5 and ζ_6 is equal zero. The first two terms in the expression contribute only to the response amplitude in the line-of-sight region based on the source position as $\theta'|\pi|$. The third term of the sum contributes to the observed response between $0 \le \theta \le \pi - \theta'$ when the angular position of the source $\theta' \le \pi$. As such to the observer satisfies the Neumann condition on the wedge at $\theta = 0$. where the effective image location is at $(r', -\theta')$. The forth term of the sum contributes to the observed response between $\pi(2\beta - 1) - \theta' < \theta < \beta\pi$ when the angular position of the source $\theta' > (\beta - 1)\pi$. As such to the observer satisfies the Neumann condition on the wedge boundary at $\theta = \beta\pi$. In this case the effective image location is $(r', 2\beta\pi - \theta')$. It is important to note that to the observer and source for any position in space only two impulses contribute to the response in the line-of-sight region;one satisfying the Neumann condition on the respective edge and other the direct response.

For the response amplitude diffracted due to edge outside the wedge is $G_{diffracted}$ the sum enclosed as the bracketed term in the integral given in Eqn.(8) can be rewritten as

$$\sum_{m=1}^{\infty} \frac{\cos(\frac{m}{\beta}\theta)\cos(\frac{m}{\beta}\theta')\sin(\frac{m}{\beta}\pi)e^{-\frac{m}{\beta}\zeta}}{\beta\pi} = \sum_{m=0}^{1}\sum_{j=0}^{1} \frac{(-1)^j}{4\beta\pi} \left[\frac{\sin(\alpha_{mj})e^{\frac{-\zeta}{\beta}}}{1 + e^{-\frac{2\zeta}{\beta}} - 2\cos(\alpha_{mj})e^{-\frac{\zeta}{\beta}}}\right] \tag{11}$$

where $\alpha_{mj} = \frac{(\theta + (-1)^m\theta' + (-1)^j\pi)}{\beta}$. Substitution of Eqn.(11) into Eqn.(8) yields diffracted response.

$$G_{diffracted} = -\int_{0}^{\infty} \frac{e^{ikR^*(\zeta) - \frac{\zeta}{\beta}}}{8\beta\pi^2 R^*(\zeta)} \sum_{m=0}^{1}\sum_{j=0}^{1}(-1)^j \left[\frac{\sin(\alpha_{mj})}{\cosh(\frac{\zeta}{\beta}) - \cos(\alpha_{mj})}\right] d\zeta \tag{12}$$

where $R^*(\zeta) = \sqrt{r^2 + r'^2 + 2rr' \cosh\zeta + z^2}$. The sum do not involve any pole on the real line and hence the exponential decay of the integrand make the integral well suited for numerical evaluation.

Therefore, the final solution for the scattered response is

$$G_\omega = \sum_{n=1}^{4} \frac{e^{ikR(\zeta_n)}}{4\pi R(\zeta_n)} \left[u(\zeta_n) - u(\zeta_n - \pi)\right] + \frac{e^{ikR(0)}}{4\pi R(0)} \left[\frac{\delta_{\theta+\theta'}}{2} + \frac{\delta_{2\pi\beta-\theta-\theta'}}{2}\right] - \int_0^\infty \frac{e^{ikR^*(\zeta)-\frac{\zeta}{\beta}}}{8\beta\pi^2 R^*(\zeta)} \sum_{m=0}^{1}\sum_{j=0}^{1}(-1)^j \left[\frac{\sin(\alpha_{mj})}{\cosh(\frac{\zeta}{\beta}) - \cos(\alpha_{mj})}\right] d\zeta \tag{13}$$

where (ζ_1,ζ_2,ζ_3, ζ_4) is equal to ($\theta - \theta'$, $\theta' - \theta$, $\theta + \theta'$,$2\pi\beta - \theta - \theta'$), $R(\zeta) = \sqrt{r^2 + r'^2 - 2rr'\cos\zeta + z^2}$, $R^*(\zeta) = \sqrt{r^2 + r'^2 + 2rr'\cosh\zeta + z^2}$ and $\alpha_{mj} = \frac{(\theta+(-1)^m\theta'+(-1)^j\pi)}{\beta}$. This simple representation of scattered response in terms of finite impulses contributing to the response in line-of-sight region and closed form integral reduces the complexity to evaluate the time dependent solution. Therefore, the impulse response due to rigid wedge is evaluated by taking inverse Fourier transform in time of Eqn.(13). The solution obtained as impulses representing the acoutic delay due to the rigid wedge and transmitted response amplitude. The inverse Fourier transform of $G_{diffracted}$ yields an integral comprising of $\delta(t - \frac{R^*(\zeta)}{c})$. So one can change the integration variable ζ to t and also replacing $d\zeta$ with $\frac{c\ R^*(\zeta)}{rr' sinh(\zeta)}dt$ and the integration limit as $\frac{R^*(0)}{c}$.

$$p = \sum_{n=1}^{4} \frac{\delta(t - \frac{R(\zeta_n)}{c})}{4\pi R(\zeta_n)} \left[u(\zeta_n) - u(\zeta_n - \pi)\right] + \frac{\delta(t - \frac{R(0)}{c})}{4\pi R(0)} \left[\frac{\delta_{\theta+\theta'}}{2} + \frac{\delta_{2\pi\beta-(\theta+\theta')}}{2}\right] - \frac{c}{8\beta\pi^2 rr' \sinh(\zeta)} \sum_{m=0}^{1}\sum_{j=0}^{1}(-1)^j \left[\frac{\sin(\alpha_{mj})}{\cosh(\frac{\zeta}{\beta}) - \cos(\alpha_{mj})}\right] u(t - \frac{R^*(0)}{c}) \tag{14}$$

where $\zeta = \cosh^{-1}(\frac{(ct)^2-(r^2+r'^2+z^2)}{2rr'})$.

IV. RESULTS

To validate the aforementioned approach this section will focus on numercial analysis of the frequency response and the modal solution. Two hundred number of terms defining the series in the modal solution to provide sufficient information for scattered response is chosen by examining the invariance of the solution with respect to the modal terms. Also in the integral in Eqn.(8) the integrand has a exponential of $\zeta\frac{m}{\beta}$ which for a five fold decay leads the limit of the integral as $0 < \zeta < \frac{5m}{\beta}$. For evaluating definite integral in the modal solution and also in the modified frequency response an adaptive 61 point Gauss-Kronard quadrature integrator is utilized.

The results shown in Fig. 2 focuses on the effective diffraction in relative range of observation distance with respect to the wavelength. The dominance of the diffraction for a short range and low frequency scattering is shown in Fig.2a for k=0.01π,r=2,r'=5.0,θ'=$\frac{\pi}{4}$,$\beta = \frac{7}{4}$. It indicates the total scattered response in the line of sight region is greater than that in the non line of sight region. Fig. 2b shows the response amplitude for moderate to higher frequency and long range scattering as k=10π,r=10,r'=5.0,θ'=$\frac{\pi}{4}$,$\beta = \frac{7}{4}$. The response amplitude in the shadow region is -80 dB less than that in the line of sight region. It indicates that the both form of solution have close agreement where dotted line represents the modified exact solution and solid line indicates the modal solution. The error is in order of 10^{-2}dB. It is important to note that the aforementioned approach results a direct frequency response and which can easily be transformed to time domain solution using inverse Fourier transform. Which because of $\sinh\zeta$ function in the denominator may lead to analytical complexity to obtain frequency response from the impulse response.

The asymptotic solution of scattered response is evaluated numerically for $kr >> 1$. Fig. 4 shows the decay of the diffracted response in the range of $0.01 < r < 500$ and $k = 30\pi$ for fixed source position and wedge angle as $r' = 10$,$\theta' = \frac{\pi}{4}$,$\beta = \frac{\pi}{4}$,$\theta = \frac{3\pi}{2}$ It indicates in the far range the diffracted response is close to order of 10^{-5} which also satisfies the principle of geomtertic optics for $\lambda \rightarrow 0$ diffracted response will be almost zero. Therefore, the asymptotic solution replaces the need stationary phase approximation.

V. SUMMARY

The exact solution for scattered response from a rigid wedge having re-entrant geometry is equivalent to six impulses representing the response in the LOS region and a series of four terms contributing to diffracted response. The new expression is numerically fast converging solution. The simplified solution contributes for a simple evaluation of scattered response at short range low and intermediate frequencies.

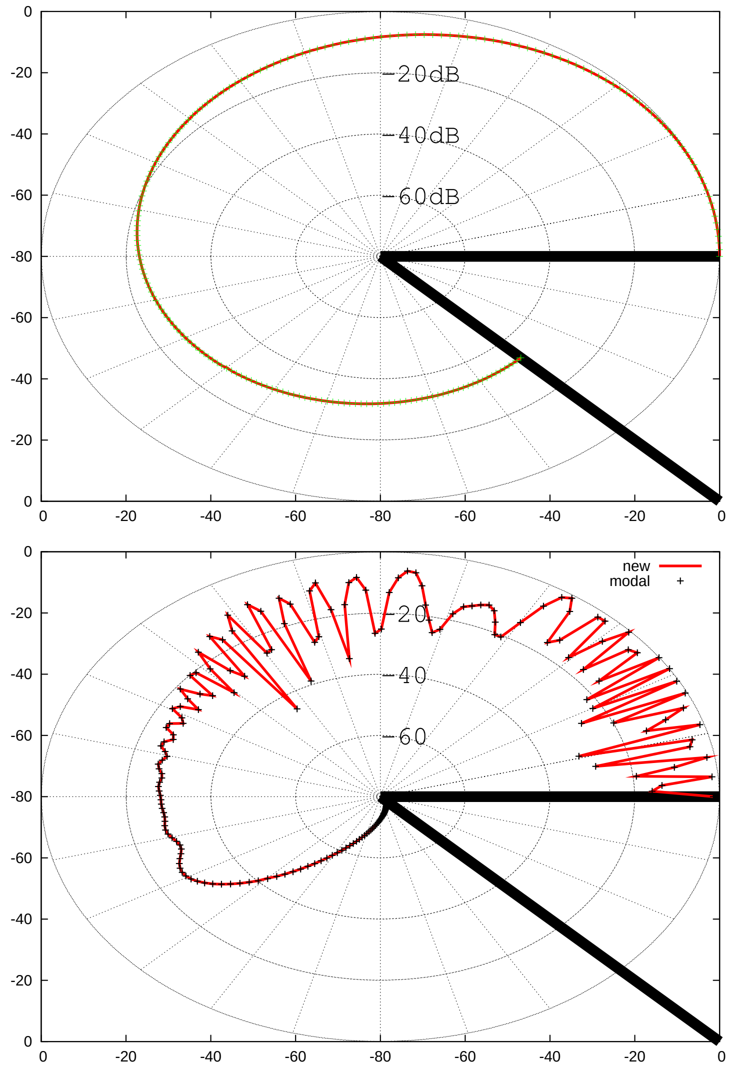

FIG. 2. a) Scattering for Low frequency b) Scattering Moderate frequency

VI. ACKNOWLEDGMENT

The authors wish to acknowledge the support of Professor Charles Thompson and Kavitha Chandra at University of Massachusetts at Lowell.

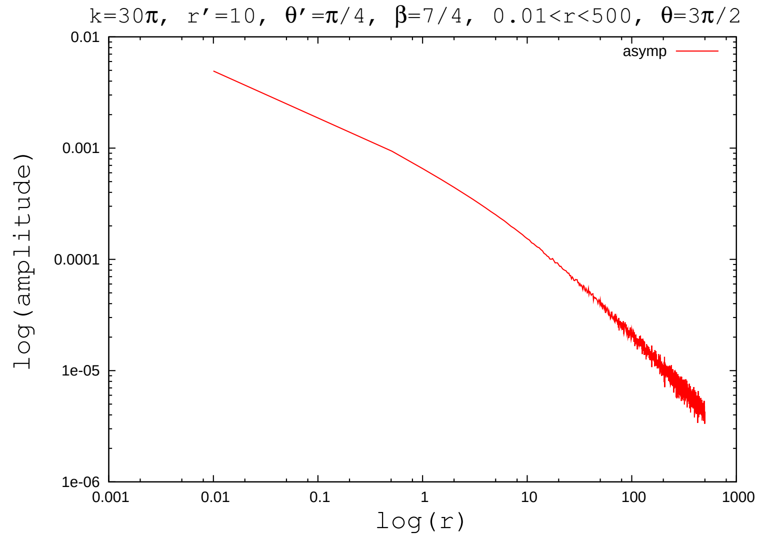

FIG. 3. Asymptotic solution of diffracted response.

References

[1] C. Huygens, *Treatise on Light* , (Macmillan and Co. Limited, 1912).

[2] A. Fresnel, *Memoir on the Diffraction of the light* , (American Book Company)pp.79-145.

[3] J.B Keller, *"Geometrical theory of Diffraction,"* Journ. optical. Soc. Am.,52 (**2**), 116-130 (1962)

[4] Arnold Sommerfeld, *Mathematical Theory of Diffraction*, (Birkhauser, 2004) pp. 9-13.

[5] H.S T.J. I'A Bromwich, *"Diffraction of waves by a wedge, "* Proc. Lond. Math. Soc.,14, 450-463 (1915)

[6] H. G Garnir, *"Fonctions De Green de l'operture metaharmonique pour les problems de Dirichlet et de Neumann poses dans un angle ou un diedre, "* Bull.Soc. Roy. Sci. Liege,14, 119-162 (1952)

[7] M.A Biot and I.Tolstoy, *"Formulation of wave propagation in infinite Media by Normal Coordinates with an application to diffraction, "* J. Acoust. Soc. Am.,3 (**29**), 381-392 (1957)

[8] F. Oberhettinger, *"On the propagation of pulses I. Diffraction of pulses by wedges, "* Zeitschrrift fur Physik,146, 423-435 (1956)

[9] Jon. H Thompson, *"Closed solutions for wedge diffraction, "* Soc. Indus. and Applied Math.,22 (**2**), 300-306 (1972)

[10] A.D. Rawlins, *"A Green's function for diffraction by a rational wedge,"* Math. Proc. Camb. Phil. Soc., (**105**), 185-192 (1989)

[11] A.D. Rawlins, *"A note on point source diffraction by a wedge,"* J. Applied. Mathematics., (**1**), 85-89 (2004),Eqn. (2.12)

[12] M.J. Buckingham, *"Theory of acoustic radiation in corners with homogeneous and mixed perfectly reflecting boundaries,"* J. Acoust. Soc. Am.,89, 2273-2291 (1989)

[13] G.Watson *A. Treatise of the Theory of Bessel functions*,(Cambridge Press,1958) pp. 75

[14] Erdelyi, Magnus, Oberhettinger and Tricomi *Bateman Manuscript project vol. II*,(MacGraw-Hill,1954) pp. 9, Eqn.(25).

[15] Erdelyi, Magnus, Oberhettinger and Tricomi *Bateman Manuscript project vol. II*,(MacGraw-Hill,1954) pp. 53, Eqn.(35).

[16] Load Kelvin, *Partial Differential Equations in Physics*, (Academic Press, 1949) pp. 79-82.

[17] Arnold Sommerfeld, *Partial Differential Equations in Physics*, (Academic Press, 1949) pp. 79-82.

List of Figures

FIG. 1 Schematic of a rigid wedge 4

FIG. 2 a) Scattering for Low frequency b) Scattering Moderate frequency 10

FIG. 3 Asymptotic solution of diffracted response. 11