Linear Tracking Control for Small-Scale Unmanned Helicopters

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Abstract—This paper presents a model-based tracking control design for small-scale unmanned helicopters. The design objective is for the helicopter to track predefined inertial position (or velocity) and heading reference trajectories. The controller is based on a nominal linear state-space model that successfully captures the small-scale helicopter coupled multivariable dynamics. The flight controller is composed of two feedback loops that regulate the tracking error of the longitudinal-lateral and heading-heave motion of the helicopter, respectively. The tracking error is determined by constructing a reference state generator based on the linear helicopter model and the reference outputs. The state generator can be systematically designed using the backstepping approach for systems in feedback form and by applying a physically meaningful approximation to the linear helicopter model. The controller performance is successfully tested using a realistic flight simulator.

Index Terms—Backstepping control, differential flatness, tracking control, trajectory generation, unmanned helicopters.

I. INTRODUCTION

UNMANNED helicopters have significant advantages over fixed-wing unmanned aerial vehicles (UAVs) because they takeoff and land vertically, they do not require a runway and they have the ability to hover and fly in low altitudes. However, helicopters are multivariable, underactuated, and highly nonlinear systems with significant inter-axis dynamic coupling. Furthermore, helicopters are considered to be much more unstable than fixed-wing aircraft, and constant control action must be sustained at all times. Therefore, it is obvious that the helicopter characteristics impose very challenging obstacles to the controller design problem.

In recent years there is considerable research related to the helicopter flight control problem. Early experimental results indicated that classical control techniques using single-input-single-output (SISO) feedback loops for each input exhibit moderate performance since they are unable to account for the coupled multivariable nature of the helicopter dynamics [2]. The majority of linear flight controllers that have been applied to autonomous helicopter platforms, are based on the $H_\infty$ approach. The advantages of the $H_\infty$ approach for the development of flight controllers for air vehicles are reported in [3]. An $H_\infty$ static output feedback controller design was proposed in [4] for the stabilization of a small-scale helicopter at hover. The output feedback approach allowed for the design of multivariable feedback loops using a reduced set of states minimizing the controller’s order. Promising flight results have been obtained in the work reported in [5] and [6]. This approach applied a blending of multivariable $H_\infty$ loop shaping control techniques and system identification for the development of the flight control system. An interesting comparative study between several controller designs is given in [7] and [8]. The flight validation indicated that in the multivariable design case it is preferable to design multiple feedback loops that correspond to decoupled subsystems of the helicopter dynamics rather than designing the controller for the complete helicopter dynamics.

Flight controllers based on modern control theory are model based since the controller architecture depends on the dynamic description of the helicopter. In most case studies that exist in the literature, the proposed designs are developed based on specific small-scale helicopter platforms. The dependence of the analysis to a particular platform is attributed to the lack of a generic nominal helicopter model that is capable of encapsulating the dynamic behavior of a large family of small-scale helicopters. The modeling approach introduced in [9] produced the first practical and yet physically consistent model structure for small-scale helicopters that has been successfully applied in many autonomous flight control applications. Due to the lack of a consistent helicopter model, most existing flight controllers are simply capable of regulating the helicopter to hover or reduced accuracy position tracking. In these cases the position tracking is typically achieved by simple feedback terms of the “velocity” error.

This paper presents a high accuracy linear tracking controller for small scale helicopters. The main objective is for the helicopter to track predefined inertial position (or velocity) and yaw reference trajectories. The controller design is based on the structure of the helicopter linear dynamic model proposed in [9]. The main novelty of this design is its ability to pass the intuitive notion of helicopter manned piloting to the mathematical derivation of the controller. This is achieved by viewing the helicopter dynamics as two subsystems, which represent the longitudinal-lateral and heading-heave motion of the helicopter, respectively. The overall control law is decomposed into two feed-
back loops that are responsible for regulating the tracking error of each subsystem. This approach achieves a more distinct effect of the helicopter inputs to the state variables of the two subsystems, compared to the case that full feedback is applied by each input.

The tracking error is determined by constructing a desired state generator-based on the helicopter model and the reference outputs. By disregarding the effect of the forces produced by the flapping motion of the main rotor, the approximated subsystems are in feedback form and, therefore, differentially flat. The differential flatness property of the approximated subsystems guarantees the existence of a desired state vector and control input such that when the helicopter’s state asymptotically converge to this desired state the output tracking goal is achieved. The overall control law is a superposition of the desired input and an output feedback component of the state error.

The output feedback component can be chosen by any design that exists in the literature. The design also allows for scheduling of multiple similar controllers based on linear models of the same structure. This work blends together concepts of nonlinear helicopter control such as differential flatness, backstepping control, and tracking control of systems in feedback form, into a simpler and more intuitive linear design. The controller performance was successfully tested in X-Plane, a commercial flight simulator which is a very good indicator of the applicability of this approach to a real flight situation. The simulation results indicate that the combination of the trajectory generator and the separation of the control law to two distinct feedback loops achieved superior tracking performance for a wide range of operating modes without the necessity of identifying multiple linear models in different operating conditions.

This paper is organized as follows. Section II presents the linear helicopter model. Section III presents the outline of the controller design and the decomposition of the helicopter dynamics to two interconnected subsystems. Section IV gives a detailed derivation of the two feedback control laws for the longitudinal-lateral and heading-heave subsystems, as well as the stability of the complete helicopter dynamics for the velocity tracking problem. Section V shows how the controller can be modified for the helicopter to track position reference trajectories. In Section VI the flight results obtained from X-Plane simulator are presented. Finally, concluding remarks are given in Section VII.

II. HELICOPTER LINEAR MODEL

The controller design should be applicable to most small-scale helicopters. This claim requires the adoption of a nominal linear dynamic model structure, which is capable of capturing the dynamic behavior of a large family of small-scale helicopters. A suitable solution to this requirement is the use of the linear parametric model developed in [9].

The linearized dynamic model proposed in [9] has been successfully adopted for control applications in a large number of small-scale helicopters of different sizes and specifications [4], [10]–[13]. These experimental applications indicate that the modeling approach proposed in [9] provides a generalized and physically sound solution for developing practical linear models for small-scale helicopters.

The helicopter motion variables are expressed with respect to a body-fixed reference frame defined as $(O_B, \mathbf{\tilde{r}}_B, \mathbf{\tilde{J}}_B, \mathbf{k}_B)$, where the center $O_B$ is located at the center of gravity (CG) of the helicopter. The directions of the body-fixed frame orthonormal vectors $(\mathbf{i}_B, \mathbf{j}_B, \mathbf{k}_B)$ are shown in Fig. 1.

The helicopter’s linear and angular velocity vectors, with respect to the body-fixed frame, are denoted by $\mathbf{\dot{v}}^B = [u \ v \ w]^T$ and $\mathbf{\omega}^B = [p \ q \ r]^T$, respectively. The helicopter attitude is expressed by the roll ($\phi$), pitch ($\theta$) and yaw ($\psi$) angles. The helicopter motion variables are shown in Fig. 1.

There are four control commands associated with helicopter piloting. The control input is defined as $u_c = [u_{\text{kn}} \ u_{\text{lat}} \ u_{\text{col}} \ u_{\text{ped}}]^T$, where $u_{\text{col}}$ and $u_{\text{ped}}$ are the collective controls of the main and tail rotor, respectively. The collective commands control the magnitude of the main and tail rotor thrust. The other two control commands $u_{\text{knv}}$ and $u_{\text{lat}}$ are the cyclic controls of the helicopter, which control the inclination of the tip-path-plane (TPP) on the longitudinal and lateral direction. The TPP is the plane in which the tips of the blades lie and it is used to provide a simplified representation of all the rotor blades.

The TPP is characterized by two angles, $a$ and $b$, which represent the tilt of the TPP at the longitudinal and lateral axis, respectively. The inclination of the TPP can be seen in Fig. 1. By controlling the inclination of the TPP the pilot indirectly controls the helicopter moments. The TPP is itself a dynamic system, which is coupled with the fuselage dynamics.

The adopted linear model represents the dynamic response of the helicopter perturbed state vector from the reference flight condition. In this case, the reference operating condition is hover. The linear state space model is described by

$$\dot{x} = Ax + Bu_c \quad (1)$$

where the state vector is given by

$$x = [u \ v \ \theta \ \phi \ q \ p \ a \ b \ w \ r \ \psi]^T. \quad (2)$$

The entries of the matrices $A \in \mathbb{R}^{11 \times 11}$ and $B \in \mathbb{R}^{11 \times 4}$, based on [9], are given in Table I. The term $g$ denotes the gravitational constant while $\tau_f$ is the main rotor’s time constant. According to [9] the stability derivatives $X_a$ and $Y_b$ are constrained to the
values \(-g\) and \(g\), respectively, due to the equality of the helicopter’s weight and main rotor’s thrust magnitudes at the trim operating condition.

The overall dynamics constitute a coupled linear system of the helicopter motion variables and the main rotor flapping dynamics. The order of the above model can be increased by including the dynamics of the stabilizer bar and the yaw damping system. These two subsystems provide additional damping to the angular velocity dynamics. Since they constitute additional feedback sources of the angular dynamics, their presence in the state space system does not influence the controller design. Therefore, their effect has been omitted from the helicopter model.

### III. CONTROLLER DESIGN OUTLINE

The controller’s ultimate objective is to autonomously track predefined bounded position and heading reference trajectories. The linear model given in (1) does not include the helicopter position dynamics. Therefore, the controller design starts with the tracking problem of a reference translational velocity and heading profile. Integration of the position tracking with the control problem follows. The initial output vector of interest of the helicopter is denoted by \(y_r = [u_r \ v_r \ w_r \ \psi_r]^T\). The primary design task is for the helicopter to track the reference output \(y_r = [u_r \ v_r \ w_r \ \psi_r]^T\). The measured states, available for feedback, are given by the vector

\[
y_m = [u \ v \ w \ p \ q \ r \ \theta \ \phi \ \psi]^T = C_m x \quad (3)
\]

where \(C_m \in \mathbb{R}^{9 \times 11}\) is a matrix of obvious entries. In real life applications, only the helicopter motion state variables can be directly measured. The motion variables can be obtained by appropriate fusion and filtering of common sensors such as a global positioning system (GPS), an inertia measurement unit (IMU), accelerometer, and a magnetic compass. However, the flapping angles are typically absent from the available measurements. When the controller’s feedback information is restricted only to a subset of the state variables, the problem is classified as output feedback. When \(y_m = x\) this results in full state feedback.

A simple tracking design is adopted which is mathematically consistent and well suited to the specific problem under consideration. The first part of the design involves determining a desired state vector \(x_d\) that is composed only by the components of the reference output vector \(y_r\) and their higher derivatives. Denote \(\varepsilon = x - x_d\) the error between the actual helicopter state and its desired value. The desired vector \(x_d\) should be chosen in such a way that, given

\[
\lim_{t \to \infty} ||\varepsilon(t)|| = 0 \quad \text{then} \quad \lim_{t \to \infty} ||y_r(t) - y_r(t)|| = 0. \quad (4)
\]

The contribution of the proposed design is the development of a simple recursive procedure for deriving the pair \((x_d, u_d^f)\) that satisfies (4) and also

\[
x_d = Ax_d + Bu_d^f. \quad (5)
\]

The choice of the pair \((x_d, u_d^f)\) is based on the backstepping design methodology. Backstepping provides a systematic methodology for the output tracking problem of systems in feedback form. Due to the presence of the stability derivatives \(X_a\) and \(Y_b\), the system in (1) does not belong to this class of systems. A common simplification practice, presented in [14], is to neglect the effect of the lateral and longitudinal forces produced by the flapping angles. These parasitic forces have a minimal effect on the translational dynamics compared to the propulsion forces produced by the stability derivatives \(X_a\) and \(Y_b\) (in (1) are denoted by \(-g\) and \(g\), respectively). This assumption is physically meaningful and results into a linear system in feedback form.

Systems of strict feedback form are feedback linearizable and, therefore, differentially flat. A system is called differentially flat when all the state and input components may be expressed in terms of output functions (called flat outputs) and their higher derivatives [15, 16].

Having defined the desired state \(x_d\) and control vector \(u_d^f\), the controller signal is constructed by the following superposition:

\[
u_c = u_d^f + u_d^{fb} \quad (6)
\]

where \(u_d^{fb}\) is a feedback control law. Then, the error dynamics take the form

\[
\dot{\varepsilon} = A\varepsilon + B u_d^{fb}. \quad (7)
\]

The second control component \(u_d^{fb}\) may be chosen using a variety of output feedback techniques, such that the error \(\varepsilon\) is rendered globally asymptotically stable (GAS).
The controller design requires that the following assumptions hold. At this stage, a preliminary control action is introduced for the input vectors $u_l$, $u_{hh}$ that normalizes the $B_l$, and $B_{hh}$ matrices, respectively. Hence

$$u_l = (B^n_l)^{-1}v_l, \quad u_{hh} = (B^n_{hh})^{-1}v_{hh}$$  \hspace{1cm} (14)$$

and $v_l$, $v_{hh}$ are control vectors to be determined. Based on Assumption 2 the above inverse matrices are nonsingular. Substituting the above preliminary control actions the two subsystems of (8) and (11), become

$$x_l = A_l x_l + B_l v_l \hspace{1cm} (16)$$
$$x_{hh} = A_{hh} x_{hh} + B_{hh} v_{hh} + D_{hh} x_l \hspace{1cm} (17)$$

where $B_l = [0_{2x6} I_2]^T$ and $B_{hh} = [0_{2x1} I_2]^T$. The initial system is now viewed as two interconnected subsystems in cascade form. The backstepping design is performed independently for each subsystem resulting in the cascaded error dynamics of the helicopter.

The controller structure requires designing two independent feedback loops for each subsystem. This approach results in a mathematically consistent and rigorous methodology, which reflects the intuitive flight notion. The longitudinal-lateral motion is regulated independently from the heading and vertical motion of the helicopter. The same decomposition of the helicopter dynamics is also reported in [7]. The stability analysis of the controller design is given in detail in the following section.

### IV. VELOCITY AND HEADING TRACKING CONTROLLER DESIGN

This section provides details for designing the controller for velocity and heading tracking of the helicopter. Based on the analysis of this section, the position tracking design follows. The control problem is focused on the design of two feedback loops for each subsystem. After the introduction of the two feedback loops the stability analysis of the overall system dynamics is presented.

#### A. Longitudinal-Lateral Dynamics

The helicopter longitudinal and lateral motion are not directly controlled through the cyclic inputs but rather via a sequence of intermediate steps. The cyclic inputs produce pitch and roll moments to the helicopter fuselage. These moments result in a change of the pitch and roll attitude angles. The attitude change results in the tilt of the helicopter main rotor disc. By tilting the
(rotor disc) the main rotor thrust is also tilted to produce the necessary propulsion forces for lateral and longitudinal motion. As indicated in Section III, the effect of the translational forces produced by the flapping motion of the main rotor is parasitic and negligible compared to the main source of propulsion, which are the forces produced by the roll and pitch attitude change of the fuselage.

By neglecting the effect of the parameters $X_a$ and $Y_b$, the longitudinal-lateral dynamics have a strict feedback form. The complete description of the longitudinal-lateral subsystem is given by

$$
\begin{align*}
\dot{v}_u &= A_{ll}^f v_u + B_{ll} v_{\text{int}} \\
v_{\text{int}} &= C_{ll} x_{ll} \\
v_{ll} &= C_{ll}^f x_u
\end{align*}
$$

where

$$
\begin{align*}
x_{ll} &= [u \ v \ \dot{\theta} \ \dot{\phi} \ p \ \alpha \ b]^T \\
v_{ll} &= [v_{\text{lon}} \ v_{\text{lat}}]^T \\
v_{\text{int}} &= [u \ v]^T.
\end{align*}
$$

In the above equations $v_{ll}$ is the measurement vector available for feedback, $y_{ll}$ is the output of the subsystem to be controlled and $C_{ll} \in \mathbb{R}^{2 \times 8}, C_{ll}^f \in \mathbb{R}^{6 \times 8}$ are matrices of obvious entries. The reference output vector is $y_{ll}^r = [u_r \ v_r]^T$. The matrix $A_{ll}^f$ is identical to $A_{ll}$ with the only difference that the stability derivatives $X_a$ and $Y_b$ are omitted. The interconnection of the approximated longitudinal-lateral subsystem is shown in Fig. 3.

The first goal of the controller design for this subsystem is to determine a desired state vector $x_{ll}^d$ and a desired control input $v_{ll}^d$ with both of them being functions of the $y_{ll}^r$ components and their higher derivatives, such that for the error $\epsilon_{ll} = x_{ll} - x_{ll}^d$ given that

$$
\lim_{t \to \infty} ||\epsilon_{ll}(t)|| = 0 \quad \text{then} \quad \lim_{t \to \infty} ||y_{ll}(t) - y_{ll}^r(t)|| = 0.
$$

(20)

To do so, the control law of this subsystem is obtained by the following superposition:

$$
v_u = v_{ll} + v_{ll}^f = [v_{ll}^d \ v_{ll}^f].
$$

where $v_{ll}^f$ is a feedback control law to be determined. The initial task is to select the pair $(x_{ll}^d, v_{ll}^f)$ such that they satisfy the requirement of (20) and also

$$
\dot{x}_{ll}^d = A_{ll}^f x_{ll}^d + B_{ll} v_{ll}^d.
$$

(22)

For the derivation of the desired state vector $x_{ll}^d$ and control input $v_{ll}^d$ a recursive procedure based on the backstepping methodology is followed such that (20) and (22) are satisfied. The applicability of this approach is based on the fact that the longitudinal-lateral subsystem is differentially flat. Therefore, the derivation of the desired state and the nominal desired input based on the reference output is feasible.

Derivation of the error dynamics and the selection of the desired states and inputs occurs simultaneously. The basic idea of the recursive procedure is to start from the top state equations of the subsystem and gradually derive the desired state variables and the error dynamics of each level by moving downwards in each step. In each step the desired values of the state variables of lower levels are chosen in such a way that they cancel out the desired values of state variables of higher levels. Notation wise, from this point forward, denote by $e_o$ the error of the variable $\alpha$ minus its desired value $\alpha_d$.

The procedure begins by deriving the error dynamics of the translational velocity variables. Therefore

$$
\dot{e}_u = \dot{u} - \dot{u}_d = -\dot{u}_d + X_u (e_u + u_d) - g(e_\theta + \theta_d). \quad \text{For} \quad \epsilon_{e_u} = u - u_d.
$$

(23)

$$
\dot{e}_v = \dot{v} - \dot{v}_d = -\dot{v}_d + Y_v (e_v + v_d) + g(e_\phi + \phi_d).
$$

(24)

The desired pitch and roll angles are chosen such that they cancel out the values $\phi_r$, $\phi_r$ and $\dot{\phi}_r$, $\dot{\phi}_r$, respectively. More precisely

$$
\theta_d = \frac{1}{g} [\dot{u}_r - X_u u_r], \quad \phi_d = \frac{1}{g} [\dot{v}_r - Y_v v_r].
$$

(25)

It is apparent that the desired angles of (25) are functions of only the $y_{ll}^r$ vector components and their first derivatives. With the above choice of the desired roll and pitch angles, the translational velocity error dynamics become

$$
\dot{e}_u = X_u e_u - g \epsilon_\theta \quad \dot{e}_v = Y_v e_v + g \epsilon_\phi.
$$

(26)

The attitude angles error dynamics are

$$
\dot{e}_\theta = -\dot{\theta}_d + q_d + e_\theta \quad \dot{e}_\phi = -\dot{\phi}_d + p_d + e_\phi.
$$

(27)

The desired values of the pitch and roll angular velocities are chosen such that they cancel out the effect of $\dot{\theta}_d$ and $\dot{\phi}_d$. Therefore

$$
\dot{q}_d = \dot{\theta}_d \quad \dot{p}_d = \dot{\phi}_d.
$$

(28)

The roll and pitch attitude error dynamics become

$$
\dot{e}_\theta = e_q \quad \dot{e}_\phi = e_p.
$$

(29)
Similarly, the angular velocity error dynamics are
\[ \dot{\epsilon}_q = -\dot{\theta}_d + M_a \dot{u}_d + M_v \dot{v}_d + M_\alpha \dot{\alpha}_d + M_e \epsilon_a + M_\epsilon \epsilon_v + M_\eta \epsilon_\eta \]  
(30)
\[ \dot{\epsilon}_p = -\dot{\phi}_d + L_a \dot{u}_d + L_v \dot{v}_d + L_\alpha \dot{\alpha}_d + L_e \epsilon_a + L_\epsilon \epsilon_v + L_\eta \epsilon_\eta . \]  
(31)

The values of the desired flapping angles \( \alpha_d \) and \( \beta_d \) are chosen as
\[ \alpha_d = \frac{1}{M_a} [\dot{u}_d - M_a \dot{u}_d - M_v \dot{v}_d] \]  
(32)
\[ \beta_d = \frac{1}{L_\alpha} [\dot{u}_d - L_\alpha \dot{u}_d - L_v \dot{v}_d] . \]  
(33)

Hence, the angular error velocity dynamics, become
\[ \dot{\epsilon}_q = M_a \epsilon_a + M_v \epsilon_v + M_e \epsilon_a \]  
(34)
\[ \dot{\epsilon}_p = L_\alpha \epsilon_a + L_v \epsilon_v + L_\epsilon \epsilon_v . \]  
(35)

Finally, the flapping angles error dynamics, are
\[ \begin{align*}
\dot{\epsilon}_a &= -\dot{\alpha}_d - q_d - \frac{1}{\tau_f} \epsilon_a + A_\theta \dot{h}_d \\
&- \frac{1}{\tau_f} \epsilon_v - A_\beta \epsilon_v + \epsilon_{km}^d + \epsilon_{km}^f \\
\dot{\epsilon}_\beta &= -\dot{\beta}_d - p_d - \frac{1}{\tau_f} \epsilon_v + B_\alpha \epsilon_a + \epsilon_{pt}^d + \epsilon_{pt}^f . 
\end{align*} \]  
(36)

The components of the control vector \( \psi_H^d \) are chosen such that they cancel out the terms of all the desired state values and only the error state variables remain in the flapping error dynamic equations. Thus
\[ \begin{align*}
\epsilon_{km}^d &= \dot{a}_d + q_d + \frac{1}{\tau_f} \epsilon_d - A_\alpha \epsilon_d \\
\epsilon_{km}^f &= \dot{b}_d + p_d + \frac{1}{\tau_f} \epsilon_v - B_\alpha \epsilon_a . 
\end{align*} \]  
(38)

It is easy to verify that the derived pair \((\chi_H^d, \psi_H^d)\) satisfies (22). The components of \( \chi_H^d \) and \( \psi_H^d \) are composed of the reference values \( \chi_r \) and \( \psi_r \) and their higher derivatives up to the fourth order. Therefore, the components of \( \psi_H^d \) should belong to \( C^4 \). The final form of the longitudinal-lateral subsystem error dynamics is
\[ \begin{align*}
\dot{\epsilon}_u &= A_H^f \epsilon_u + \bar{B}_H \psi_H^f \\
Y_H^m &= C_H^m \epsilon_u \end{align*} \]  
(39)

where
\[ \begin{align*}
\epsilon_u &= [\epsilon_a \ \epsilon_v \ \epsilon_\theta \ \epsilon_\phi \ \epsilon_q \ \epsilon_p \ \epsilon_a \ \epsilon_\beta]^T \\
Y_H^m &= [\epsilon_a \ \epsilon_v \ \epsilon_\theta \ \epsilon_\phi \ \epsilon_q \ \epsilon_p]^T . 
\end{align*} \]  
(40)

In the above equations \( Y_H^m \) is the measurement vector of the longitudinal-lateral error subsystem. The initial tracking problem of the longitudinal and lateral dynamics has been converted to the stabilization problem of the error vector \( \epsilon_u \). The measurement vector \( Y_H^m \) does not have available all the state variables of the system (39) since the flapping angles \( \alpha \) and \( \beta \) cannot be measured. A static feedback control law is required of the form
\[ \psi_H^f = -K_H Y_H^m \]  
(41)

with \( K_H \) being a gain matrix, such that for the closed-loop system
\[ \dot{\epsilon}_u = (A_H^f - \bar{B}_H K_H C_H^m) \epsilon_u \]  
(42)

the closed-loop matrix \( A_H^f - \bar{B}_H K_H C_H^m \) is Hurwitz.

Details about the output feedback problem are given in [17] and [18]. Stabilization via output feedback can be achieved by two ways: Eigenvalue placement and in the context of the linear quadratic regulator (LQR). The eigenvalue placement approach typically requires the solution of very complicated heuristic algorithms for the calculation of the output feedback gain. For this reason, the LQR approach is preferred. The LQR controller design for LTI systems with output feedback was initially introduced in [19]. Generally, optimal control with output feedback requires the solution of coupled nonlinear matrix equations [20]. There are several iterative algorithms for the solution of the above problem. However, the most practical convergent algorithm that results in a local minimum solution is given in [20] based on [21].

### B. Heading-Heave Dynamics

The goal of this section is the design of the second control law responsible for the heading and vertical velocity tracking. The heading-heave dynamics subsystem, is summarized by the following equations:
\[ \begin{align*}
\dot{x}_{hh} &= A_{hh} x_{hh} + \bar{B}_{hh} \psi_{hh} + D_{hh} x_{hh} \\
y_{hh} &= C_{hh} x_{hh} \\
y_{hh}^m &= \chi_{hh} 
\end{align*} \]  
(43)

where
\[ \begin{align*}
x_{hh} &= [\psi \ \psi \ \tau]^T \\
\psi_{hh} &= [\psi_{\text{ped}} \ \psi_{\text{car}}]^T \]  
(44)

In the above equations, \( y_{hh} \) is the output vector to be controlled, \( y_{hh}^m \) is the measurement vector and \( C_{hh} \subseteq \mathbb{R}^{2 \times 3} \) is a matrix of obvious entries. The reference output is denoted by \( y_{hh}^r = [\psi_r \ \psi_r \ \tau]^T . \) The heading-heave subsystem is in cascade connection with the longitudinal-lateral subsystem via the matrix \( D_{hh} \subseteq \mathbb{R}^{3 \times 8} \). The interconnection of the heading-heave subsystem dynamics is shown in Fig. 4. The design procedure is similar with the one presented in Section IV-A. The controller design requires the determination of a desired state vector \( x_{hh}^d \) and a desired nominal control input \( \psi_{hh}^f \), such that when the error \( \epsilon_{hh} = x_{hh} - x_{hh}^d \) is regulated to zero, then, the output \( y_{hh} \) of the yaw heave subsystem asymptotically tracks the reference output.
vector $y_{hh}$. The control law for the heading-heave subsystem is obtained as the following superposition:

$$v_{hh} = v_{hh}^d + v_{hh}^f = \begin{bmatrix} v_{hh}^d \\ v_{hh}^f \end{bmatrix} = \begin{bmatrix} v_{hh}^d \\ v_{hh}^f \end{bmatrix}$$ (45)

where $v_{hh}^f$ is a feedback control vector to be determined. The choice of the controller parameter $v_{hh}^d$ and the desired state vector $x_{hh}^d$ should satisfy

$$x_{hh}^d = A_{hh}x_{hh}^d + B_{hh}v_{hh}^f + D_{hh}x_{hl}^d$$ (46)

where the state vector $x_{hl}^d$ is defined in Section IV-A. The input $v_{hh}^f$ and the desired state $x_{hh}^d$, are derived by using a similar recursive backstepping procedure with the one described in Section IV-A. The choice of $v_{hh}^d$ and $x_{hh}^d$ components emerge from the inspection of the error vector $\xi_{hh} = x_{hh} - x_{hh}^d$, dynamics. The error dynamics of the heading-heave subsystem are given by

$$\dot{\xi}_h = -\xi_h + r_d + e_r$$
$$\dot{\xi}_w = -\xi_w + Z_u e_d + Z_b d_d + Z_r r_d + Z_u w_d + Z_e e_d + \xi_d + v_{hh}^d$$
$$\dot{\xi}_r = -\xi_r + N_u e_d + N_b d_d + N_r r_d + N_w e_d + N_r e_d + v_{hh}^d + v_{hh}^f$$

(47-49)

The desired angular velocity $r_d$, and the components of $v_{hh}^d$, are chosen such that they cancel out all the terms associated with the rest of the desired state variables and only the error terms remain in the heading-heave subsystem error dynamics. Thus

$$r_d = \dot{\xi}_r$$

(50)

$$v_{hh}^d = -\xi_d - N_u e_d - N_b d_d - N_r r_d$$

(51)

$$v_{hh}^d = -\xi_d - Z_u e_d - Z_b d_d - Z_r r_d$$

(52)

Based on the above choice, it is easy to verify that (46) is satisfied. The desired state vector $x_{hh}^d$ and the control input $v_{hh}^d$ are functions of the components of $y_{hh}^d$, $y_{hl}^f$ vectors and their higher derivatives. Moreover, $\xi_h$, $\xi_w$, and $\xi_r$ should belong to $C^2$ and $C^3$, respectively. The dependence of $y_{hh}^d$ stems from the interconnection of the two subsystems through the matrix $D_{hh}$. Using the equations given in (50)–(52), the error dynamics of the heading-heave subsystem become

$$\dot{\xi}_{hh} = A_{hh}\xi_{hh} + B_{hh}v_{hh}^f + D_{hh}\xi_{hl}$$

(53)

where $\xi_{hh} = [\xi_h, \xi_r, \xi_w]_T$. In the above equations $Y_{hh}^m$ denotes the vector of available measurements. Similarly with the longitudinal-lateral subsystem, the tracking problem of $y_{hh}^f$ is converted to the regulation of $\xi_{hh}$ to zero. However, in this particular case, the full state vector of the system in (53) is available for feedback. The design objective is to determine a static feedback law $v_{hh}^f$ of the form

$$v_{hh}^f = -K_{hh}\xi_{hh}$$ (54)

where $K_{hh}$ is a gain matrix, such that the closed loop stability matrix $A_{hh}^f = A_{hh} - B_{hh}K_{hh}$ of the heading-heave error subsystem is Hurwitz. As it will be illustrated later, if this condition is satisfied, the solution of the complete error dynamics is GAS given that $A_{hh}^f$ is Hurwitz as well. For the heading-heave subsystem the choice of $K_{hh}$ is much simpler since we have full state feedback. In this case both the LQR approach and the eigenvalue placement can be easily applied.

C. Stability of the Complete System Error Dynamics

By applying the control laws $v_{hh}^f$ and $v_{hl}^f$, the complete error system dynamics take the form

$$\begin{bmatrix} \dot{\xi}_{hh} \\ \dot{\xi}_{hl} \end{bmatrix} = \begin{bmatrix} (A_{hh} - B_{hh}K_{hh}) & D_{hh} \\ 0_{8\times3} & (A_{hl}^f - B_{hl}K_{hl}C_{hl}^m) \end{bmatrix} \begin{bmatrix} \xi_{hh} \\ \xi_{hl} \end{bmatrix}$$ (55)

The stability of the complete error dynamics system given in (55), is specified by the following Theorem.

Theorem 1: Given that the feedback gains $K_{hl}$ and $K_{hh}$ are selected such that the matrices $A_{hl}^f = A_{hl}^f - B_{hl}K_{hl}C_{hl}^m$ and $A_{hh}^f = A_{hh} - B_{hh}K_{hh}$ are Hurwitz, then the solution $[\xi_{hh}(t), \xi_{hl}(t)]^T$ of the complete error dynamics system of (55) is GAS.

Proof: Since the state matrix of (55) is in block triangular form, its eigenvalues satisfy the characteristic equation given by the following equality:

$$\det(A_{hh}^d - \lambda I_{3\times3}) \cdot \det(A_{hl}^f - \lambda I_{8\times8}) = 0$$ (56)

where $\det(\cdot)$ denotes the determinant of a matrix and $\lambda \in \mathbb{R}$ a nominal eigenvalue of the system of (55). From the above characteristic equation, it is obvious that the eigenvalues of the composite error system are the union of the eigenvalues of $A_{hh}^f$ and $A_{hl}^d$. Since both of these matrices are Hurwitz, then all the eigenvalues of (55) have strictly negative real parts. Therefore, the complete error dynamics system of (55) is GAS.

The above stability analysis is based on the approximate helicopter model. To investigate the stability of the actual helicopter dynamic model we consider the restrictions that are imposed to the flapping angles by the mechanical configuration of the rotor hub. For most rotor hub mechanical configurations of both full-scale and small-scale helicopters the flapping angles have typically a bound of 15° [9], [22], [23]. The resulting actual, non-approximated, helicopter error dynamics may be expressed as

$$\dot{\xi} = A(\xi) + \Delta(a, b)$$ (57)
where
\[ \epsilon = [\epsilon_{hh} \, \epsilon_{th}]^T \]
\[ \Delta(a, b) = \begin{bmatrix} 0_{2 \times 3} & g \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T \\ 0_{2 \times 6} \end{bmatrix}^T \begin{bmatrix} a \\ b \end{bmatrix} \] (58)
and
\[ A_e = \begin{bmatrix} (A_{hh} - \bar{B}_{hh}K_{hh}) \\ 0_{3 \times 3} \end{bmatrix} \begin{bmatrix} D_{hh} \\ (A_{th} - \bar{B}_hK_{th}C_{th}) \end{bmatrix}. \] (59)

The mechanical restriction to the flapping angles is described by the bound \( \|\Delta(a, b)\| \leq \delta \) with \( \delta > 0 \). It is easy to show that \( \|\Delta(a, b)\| \leq g\delta \). The stability of the actual error dynamics system given in (57), is specified by the following Theorem.

**Theorem 2:** Given that the feedback gains \( K_{th} \) and \( K_{hh} \) are selected such that the matrices \( A_{th} = A_{th}^T - \bar{B}_hK_{th}C_{th} \) and \( A_{hh} = A_{hh}^T - \bar{B}_{hh}K_{hh} \) are Hurwitz, then the solution \( [\epsilon_{hh}(t), \epsilon_{th}(t)]^T \) of the actual error dynamics system of (57) is globally uniformly ultimately bounded.

**Proof:** Since the matrix \( A_e \) is Hurwitz, based on standard results there exist a positive definite symmetric matrix \( P \in \mathbb{R}^{11 \times 11} \), such that
\[ PA_e + A_e^T P = -Q \] (60)
where \( Q \in \mathbb{R}^{11 \times 11} \) is a positive definite symmetric matrix. Consider the Lyapunov function \( V(\epsilon) = \epsilon^T P \epsilon \). The derivative of \( V(\epsilon) \) along the trajectories of the system (57) is given by
\[ \dot{V}(\epsilon) = \epsilon^T \left( PA_e + A_e^T P \right) \epsilon + \epsilon^T A_e^T P \Delta(a, b) \]
\[ = -\epsilon^T Q \epsilon + 2\epsilon^T P \Delta(a, b) \]
\[ \leq -\lambda_{\text{min}}(Q) \|\epsilon\|^2 + 2\lambda_{\text{max}}(P) g\delta \|\epsilon\| \]
where \( \lambda_{\text{min}}(\cdot) \), \( \lambda_{\text{max}}(\cdot) \) denote the minimum and maximum eigenvalues of a matrix, respectively. From the above inequality it is obvious that
\[ \dot{V}(\epsilon) < 0, \quad \forall \|\epsilon\| \geq \frac{2\lambda_{\text{max}}(P) g\delta}{\lambda_{\text{min}}(Q)}. \] (61)

Therefore the solutions of \( \epsilon(t) \) given by (57) are globally uniformly ultimately bounded.

**V. POSITION AND HEADING TRACKING**

The ultimate goal of the controller design is for the helicopter to track a predefined position trajectory of an inertial frame expressed by the reference vector \( \dot{p}_{r}^I = [\dot{p}_{r.x} \, \dot{p}_{r.y} \, \dot{p}_{r.z}]^T \). The inertial frame (Earth-fixed frame) is defined as \( F_I = \{O_I, \overrightarrow{r_I}, \hat{j}_I, \hat{k}_I\} \). A typical convention of the Earth-fixed frame is the North-East-Down system where \( \overrightarrow{r}_I \) points North, \( \hat{j}_I \) points East and \( \hat{k}_I \) points at the center of the Earth. The position error dynamics are derived by using the properties of the rotation matrix \( R(\theta) \in \text{SO}(3) \), where \( \text{SO}(3) \) denotes the Special Orthogonal group of order three. The rotation matrix is parametrized with respect to the three Euler angles and it is used to map vectors from the body-fixed frame \( F_B \) to the inertial frame denoted by \( F_I \). From standard results, using the Z-Y-X Euler angles, the rotation matrix is given by
\[ R(\theta) = \begin{bmatrix} C_\psi C_\theta & -S_\psi C_\phi + C_\phi S_\psi S_\theta & S_\phi S_\psi + C_\phi C_\psi S_\theta \\ S_\psi C_\phi + C_\phi S_\psi S_\theta & -C_\psi C_\phi + S_\phi S_\psi S_\theta & -S_\phi S_\psi + C_\phi C_\psi S_\theta \\ -S_\phi C_\theta & C_\phi S_\theta & C_\phi C_\theta \end{bmatrix}. \] (62)

The abbreviations \( C_l \) and \( S_l \) with \( l \in \mathbb{R} \) represent the trigonometric functions \( \cos(l) \) and \( \sin(l) \), respectively. For an arbitrary motion, the components of the rotation matrix are time varying. The derivative of the rotation matrix is given by
\[ \dot{R} = R \hat{\omega}^B \] (63)
where \( \hat{\omega}^B \) denotes the skew symmetric matrix of the vector \( \omega^B \).

For a vector \( c = [c_1 \, c_2 \, c_3]^T \) the skew symmetric matrix is defined as
\[ \hat{c} = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix}. \]

The position error expressed in the body-fixed frame is given by
\[ e_p^B = R^T (p^I - p_r^I). \] (64)

Using (63), the position error dynamics are given by
\[ \dot{e}_p^B = R^T (\dot{p}^I - \dot{p}_r^I) + R \hat{\omega}^B (p^I - p_r^I) = R^T (v^I - v_r^I) + (R \hat{\omega}^B)^T (p^I - p_r^I) = v^B - v_r^B + (\hat{\omega}^B)^T R^T (p^I - p_r^I) = e_p^B + \hat{\omega}^B \hat{e}_p^B = \hat{\omega}^B e_p^B = \hat{\omega}^B \hat{e}_p^B. \] (65)

In order to derive the position error dynamics the following have been considered:
\[ v_r^I = \dot{p}_r^I, \quad v^I = \dot{p}^I, \quad \hat{\omega}^B = - (\hat{\omega}^B)^T, \quad \hat{\omega}^B e_p^B = - \hat{\omega}^B \hat{e}_p^B. \] (66)

The position error dynamics are not linear since they include the nonlinear term \( \hat{\omega}^B \hat{e}_p^B \). The latter term expresses the contribution of the angular velocity to the position error dynamics.

The choice of a linear model for the representation of the helicopter dynamics is limited to a certain range of a particular operating condition. In this case, the operating condition of interest is the hover flight mode. Since the linear model of (1) is restricted within a certain range of the operating condition, the tracking problem of arbitrary position and velocity trajectories becomes dubious. However, experimental results of realistic life applications indicate that the accuracy of linear dynamic models is satisfactory enough for a relatively wide range of the flight envelope around the reference operating condition \([9, 23, 24]\). Therefore, it is assumed that the adopted linear model of (1) provides a quasi-global description of the helicopter dynamics. Linearization is also applied to the nonlinear position error dynamics, assuming that \( e_p^B \) is the perturbed value of the position error from the reference steady-state vector \( e_{p, ref}^B = [0 \, 0 \, 0]^T \). In this
case, the term $\dot{e}_{p}^{B}w^{B}$ can be disregarded since it is considered as a product of two perturbed values. This approximation adds up to all simplification assumptions that take place in order to obtain the linear dynamic model of the helicopter given in (1). Therefore, the approximated position error dynamics are given by

$$\dot{e}_{p}^{B} = e_{\psi}^{B}.$$  

(67)

The composite error system is additionally enhanced by the integral of the position and yaw error dynamics. The presence of integral terms in the control law is very beneficial in terms of robustness performance. The feedback integral components attenuate the steady state tracking error caused by potential parametric and model uncertainty. Denote by $\eta_{p}^{B} = [\eta_{x}^{B} \eta_{y}^{B} \eta_{z}^{B}]^T$ and $\eta_{\psi}$ the integral of the position and yaw error. Thus

$$\dot{\eta}_{p}^{B} = e_{p}^{B} \quad \dot{\eta}_{\psi} = e_{\psi}.$$  

(68)

The structure of the control laws for the position tracking problem will be identical to the velocity tracking case. The composite error dynamics are still separated into two subsystems corresponding to the lateral-longitudinal and heading-heave motion. From this point forward the state error of the two subsystems including the position and the position integral error is denoted by $\epsilon$. Having said that, the longitudinal-lateral dynamics are given by

$$\dot{\epsilon}_{ll} = A_{ll}\epsilon_{ll} + B_{ll}\epsilon_{ll}^{fb} \quad \gamma_{ll}^{m} = \epsilon_{ll}$$  

(69)

where

$$\epsilon_{ll} = [\eta_{x}^{B} \eta_{y}^{B} e_{x}^{B} e_{y}^{B} e_{u} e_{v} e_{\theta} e_{\phi} e_{q} e_{p} e_{w} e_{r}]^T \gamma_{ll}^{m} = [\eta_{x}^{B} \eta_{y}^{B} e_{x}^{B} e_{y}^{B} e_{u} e_{v} e_{\theta} e_{\phi} e_{q} e_{p}]^T$$  

(70)

and

$$\begin{align*}
A_{ll} &= \begin{bmatrix}
0_{4 \times 4} & I_{4 \times 4} & 0_{4 \times 6} \\
0_{8 \times 2} & 0_{8 \times 2} & A_{ll}^{R} \\
0_{8 \times 2} & 0_{8 \times 2} & A_{ll}^{I}
\end{bmatrix}, \\
B_{ll} &= \begin{bmatrix}
0_{4 \times 2} \\
0_{8 \times 2} \\
0_{8 \times 2}
\end{bmatrix},
\end{align*}$$  

(71)

In the above equations $\gamma_{ll}^{m}$ denotes the available measurements vector and $C_{ll}^{m} \in \mathbb{R}^{10 \times 12}$ a matrix of obvious entries. The heading-heave error dynamics are given by

$$\dot{\epsilon}_{hh} = A_{hh}\epsilon_{hh} + B_{hh}\epsilon_{hh}^{fb} + D_{hh}\epsilon_{ll} \quad \gamma_{hh}^{m} = \epsilon_{hh}$$  

(72)

where

$$\epsilon_{hh} = [\eta_{x}^{B} \eta_{y}^{B} e_{x}^{B} e_{y}^{B} e_{w} e_{r}]^T$$  

(73)

and

$$\begin{align*}
A_{hh} &= \begin{bmatrix}
0_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 1} \\
0_{3 \times 2} & 0_{3 \times 1} & A_{hh} \\
0_{3 \times 4} & 0_{3 \times 8} & D_{hh}
\end{bmatrix}, \\
B_{hh} &= \begin{bmatrix}
0_{3 \times 2} \\
0_{3 \times 2} \\
0_{3 \times 4}
\end{bmatrix}, \\
D_{hh} &= \begin{bmatrix}
0_{3 \times 2} & 0_{3 \times 1} \\
0_{3 \times 4} & 0_{3 \times 8} & D_{hh}
\end{bmatrix},
\end{align*}$$  

(74)

In the above equations $\gamma_{hh}^{m}$ denotes the available measurements vector. The interconnection of the new complete error dynamics subsystems is illustrated in Fig. 5. Similarly to the velocity tracking case, the control design is reduced to the calculation of two feedback gain matrices $K_{ll}$ and $K_{hh}$, such that by applying the following feedback control laws:

$$v_{ll}^{fb} = -K_{ll}\gamma_{ll}^{m}$$  

(75)

$$v_{hh}^{fb} = -K_{hh}\gamma_{hh}^{m}$$  

(76)

the closed-loop matrices $A_{ll}^{fb} = A_{ll} - B_{ll}K_{ll}\gamma_{ll}^{m}$ and $A_{hh}^{fb} = A_{hh} - B_{hh}K_{hh}$ are Hurwitz. The feedback gains may be calculated by following the analysis described in Sections IV-A and IV-B.

However, any methodology that determines the feedback gains of the laws (75) and (76) requires that the matrix pairs $(A_{ll}^{fb}, B_{ll})$ and $(A_{hh}^{fb}, B_{hh})$ are controllable. The necessary condition for controllability of the pairs $(A_{ll}^{fb}, B_{ll})$ and $(A_{hh}^{fb}, B_{hh})$ is established by the following Theorem.

Theorem 3: Given that Assumptions 1, 2 and 3 hold, then the pairs $(A_{ll}^{fb}, B_{ll})$ and $(A_{hh}^{fb}, B_{hh})$ are controllable.

Proof: Based on Assumptions 1 and 2, the pair $(A_{ll}^{fb}, B_{ll})$ is controllable. Let $T(s) = sI_{8} - A_{ll}^{fb}[B_{ll}]$, where $s \in \mathbb{R}$. From the Popov-Belevitch-Hautus (PBH) test, for every $s \in \mathbb{R}$, rank $(T(s)) = 8$. Thus, one must show that rank $(T(s)) = 12$ for every $s \in \mathbb{R}$, where $T(s) = [sI_{12} - A_{ll}^{fb}[B_{ll}]]$.

For $s \neq 0$ one has

$$\text{rank}(T(s)) = \text{rank}\begin{bmatrix}
sI_{2} & -I_{2} & 0_{2 \times 2} & 0_{2 \times 6} & 0_{4 \times 2} \\
0_{2 \times 2} & sl_{2} & -I_{2} & 0_{2 \times 6} & 0_{4 \times 2} \\
0_{8 \times 2} & 0_{8 \times 2} & sI_{8} & A_{ll}^{fb} & B_{ll}
\end{bmatrix}.$$  

(77)

Since $s \neq 0$, the first four rows are linearly independent. Therefore

$$\text{rank}(T(s)) = 4 + \text{rank}\begin{bmatrix}
sI_{8} & -A_{ll}^{fb}[B_{ll}]
\end{bmatrix} = 4 + 8 = 12.$$  

(78)

For $s = 0$ one has

$$\text{rank}(T(0)) = \text{rank}\begin{bmatrix}
0_{2 \times 2} & -I_{2} & 0_{2 \times 2} & 0_{2 \times 6} & 0_{4 \times 2} \\
0_{2 \times 2} & 0_{2 \times 2} & -I_{2} & 0_{2 \times 6} & 0_{4 \times 2} \\
0_{8 \times 2} & 0_{8 \times 2} & -A_{ll}^{fb} & B_{ll}
\end{bmatrix}.$$  

(79)
Calculate $K_{ii}$ and $K_{hh}$, such that the closed loop matrices $A_{ii}^d = A_{ii} - B_{ii} K_{ii} C_{ii}^m$ and $A_{hh}^d = A_{hh} - B_{hh} K_{hh}$ are Hurwitz.

- Given a reference position vector $p_i^r = [p_i^r, p_i^r, p_i^r, p_i^r]^T$, with respect to the inertial frame, the reference velocities values $v_i^r = [u_r, v_r, w_r]^T$, in the body-fixed frame, are given by $v_i^r = R^T p_i^r$.

- The desired state variables and control inputs for the longitudinal-lateral subsystem are

$$
\begin{align*}
\theta_d &= \frac{1}{g} [u_r - X_u u_r] \\
\phi_d &= \frac{1}{g} [v_r - Y_u v_r] \\
q_d &= \dot{\theta}_d \\
p_d &= \dot{\phi}_d \\
a_d &= \frac{1}{M_a} [q_d - M_u u_d - M_v v_d] \\
b_d &= \frac{1}{L_a} [p_d - L_u u_d - L_v v_d] \\
v_{\text{con}}^d &= \dot{a}_d + q_d + \frac{1}{\tau_f} a_d - A_h b_d \\
v_{\text{int}}^d &= \dot{b}_d + p_d + \frac{1}{\tau_f} b_d - B_a a_d.
\end{align*}
$$

- The desired state variables and control inputs for the heading-heave subsystem are

$$
\begin{align*}
r_d &= \dot{\psi}_r \\
v_{\text{ped}}^d &= \dot{r}_d - N_u v_d - N_p p_d - N_w w_d - N_r r_d \\
v_{\text{col}}^d &= \dot{w}_d - Z_a a_d - Z_b b_d - Z_r r_d - Z_w w_d.
\end{align*}
$$

- From the helicopter’s available measurements and the reference trajectory calculate $v_{\text{ii}}^d$, $v_{\text{hh}}^d$ and $Y_{\text{ii}}^m$, $Y_{\text{hh}}^m$.

- Calculate the two feedback control laws

$$
\begin{align*}
v_{\text{ii}} &= v_{\text{ii}}^d - K_{ii} Y_{\text{ii}}^m & v_{\text{hh}} &= v_{\text{hh}}^d - K_{hh} Y_{\text{hh}}^m.
\end{align*}
$$

- The final control vectors for the two subsystems are obtained by

$$
\begin{align*}
u_{\text{ii}} &= (B_{\text{ii}}^d)^{-1} v_{\text{ii}} & u_{\text{hh}} &= (B_{\text{hh}}^d)^{-1} v_{\text{hh}}.
\end{align*}
$$

The first two rows are linearly independent. Therefore

$$\text{rank}(T(0)) = 2 + \text{rank} \begin{bmatrix} -I_2 & 0_{2 \times 6} \\ -A_{\text{ii}}^d & B_{\text{ii}} \end{bmatrix}.$$}

The matrix of the right-hand side of the above equation, is square and lower triangular with nonzero elements in its main diagonal (this fact is guaranteed by Assumption 3). Hence, the rank of this matrix is 10 and $\text{rank}(T(0)) = 12$.

Therefore, it has been proved that for every $s \in \mathbb{R}$, $\text{rank}(T(s)) = 12$. So, given that the pair $(A_{\text{ii}}^d, B_{\text{ii}})$ is controllable, then the pair $(A_{\text{ii}}^d, B_{\text{ii}})$ is controllable as well. The proof for the controllability of $(A_{\text{hh}}, B_{\text{hh}})$ based on the controllability of the pair $(A_{\text{hh}}, B_{\text{hh}})$ is derived in a similar way.

By applying the control laws $v_{\text{ii}}^f = -K_{ii} Y_{\text{ii}}^m$ and $v_{\text{hh}}^f = -K_{hh} Y_{\text{hh}}^m$, the complete error system dynamics take the form

$$
\dot{\epsilon} = A_{\text{e}}^d \epsilon
$$

(77)

where

$$
\epsilon = \begin{bmatrix} \epsilon_{\text{hh}} \\ \epsilon_{\text{ii}} \end{bmatrix}, \quad A_{\text{e}}^d = \begin{bmatrix} (A_{\text{hh}} - B_{\text{hh}} K_{\text{hh}}) & D_{\text{hh}} \\ 0_{12 \times 6} & (A_{\text{ii}} - B_{\text{ii}} K_{\text{ii}} C_{\ii}^m) \end{bmatrix}.
$$

(78)

The stability of the complete error system dynamics of (77) is established by the following Theorem.

**Theorem 4:** Given that the feedback gains $K_{ii}$ and $K_{hh}$ are selected such that the matrices $A_{\text{ii}}^d = A_{\text{ii}} - B_{\text{ii}} K_{\text{ii}} C_{\ii}^m$ and $A_{\text{hh}}^d = A_{\text{hh}} - B_{\text{hh}} K_{\text{hh}}$ are Hurwitz, then the solution

**TABLE II**

OUTLINE OF THE CONTROLLER DESIGN FOR POSITION AND HEADING TRACKING

- Calculate $K_{ii}$ and $K_{hh}$, such that the closed loop matrices $A_{ii}^d = A_{ii} - B_{ii} K_{ii} C_{ii}^m$ and $A_{hh}^d = A_{hh} - B_{hh} K_{hh}$ are Hurwitz.
- Given a reference position vector $p_i^r = [p_i^r, p_i^r, p_i^r, p_i^r]^T$, with respect to the inertial frame, the reference velocities values $v_i^r = [u_r, v_r, w_r]^T$, in the body-fixed frame, are given by $v_i^r = R^T p_i^r$.
- The desired state variables and control inputs for the longitudinal-lateral subsystem are

$$
\begin{align*}
\theta_d &= \frac{1}{g} [u_r - X_u u_r] \\
\phi_d &= \frac{1}{g} [v_r - Y_u v_r] \\
q_d &= \dot{\theta}_d \\
p_d &= \dot{\phi}_d \\
a_d &= \frac{1}{M_a} [q_d - M_u u_d - M_v v_d] \\
b_d &= \frac{1}{L_a} [p_d - L_u u_d - L_v v_d] \\
v_{\text{con}}^d &= \dot{a}_d + q_d + \frac{1}{\tau_f} a_d - A_h b_d \\
v_{\text{int}}^d &= \dot{b}_d + p_d + \frac{1}{\tau_f} b_d - B_a a_d.
\end{align*}
$$

- The desired state variables and control inputs for the heading-heave subsystem are

$$
\begin{align*}
r_d &= \dot{\psi}_r \\
v_{\text{ped}}^d &= \dot{r}_d - N_u v_d - N_p p_d - N_w w_d - N_r r_d \\
v_{\text{col}}^d &= \dot{w}_d - Z_a a_d - Z_b b_d - Z_r r_d - Z_w w_d.
\end{align*}
$$

- From the helicopter’s available measurements and the reference trajectory calculate $v_{\text{ii}}^d$, $v_{\text{hh}}^d$ and $Y_{\text{ii}}^m$, $Y_{\text{hh}}^m$.
- Calculate the two feedback control laws

$$
\begin{align*}
v_{\text{ii}} &= v_{\text{ii}}^d - K_{ii} Y_{\text{ii}}^m & v_{\text{hh}} &= v_{\text{hh}}^d - K_{hh} Y_{\text{hh}}^m.
\end{align*}
$$

- The final control vectors for the two subsystems are obtained by

$$
\begin{align*}
u_{\text{ii}} &= (B_{\text{ii}}^d)^{-1} v_{\text{ii}} & u_{\text{hh}} &= (B_{\text{hh}}^d)^{-1} v_{\text{hh}}.
\end{align*}
$$

The first two rows are linearly independent. Therefore

$$\text{rank}(T(0)) = 2 + \text{rank} \begin{bmatrix} -I_2 & 0_{2 \times 6} \\ -A_{\text{ii}}^d & B_{\text{ii}} \end{bmatrix}.$$}

The matrix of the right-hand side of the above equation, is square and lower triangular with nonzero elements in its main diagonal (this fact is guaranteed by Assumption 3). Hence, the rank of this matrix is 10 and $\text{rank}(T(0)) = 12$.

Therefore, it has been proved that for every $s \in \mathbb{R}$, $\text{rank}(T(s)) = 12$. So, given that the pair $(A_{\text{ii}}^d, B_{\text{ii}})$ is controllable, then the pair $(A_{\text{ii}}^d, B_{\text{ii}})$ is controllable as well. The proof for the controllability of $(A_{\text{hh}}, B_{\text{hh}})$ based on the controllability of the pair $(A_{\text{hh}}, B_{\text{hh}})$ is derived in a similar way.

By applying the control laws $v_{\text{ii}}^f = -K_{ii} Y_{\text{ii}}^m$ and $v_{\text{hh}}^f = -K_{hh} Y_{\text{hh}}^m$, the complete error system dynamics take the form

$$
\dot{\epsilon} = A_{\text{e}}^d \epsilon
$$

(77)

where

$$
\epsilon = \begin{bmatrix} \epsilon_{\text{hh}} \\ \epsilon_{\text{ii}} \end{bmatrix}, \quad A_{\text{e}}^d = \begin{bmatrix} (A_{\text{hh}} - B_{\text{hh}} K_{\text{hh}}) & D_{\text{hh}} \\ 0_{12 \times 6} & (A_{\text{ii}} - B_{\text{ii}} K_{\text{ii}} C_{\ii}^m) \end{bmatrix}.
$$

(78)

The stability of the complete error system dynamics of (77) is established by the following Theorem.

**Theorem 4:** Given that the feedback gains $K_{ii}$ and $K_{hh}$ are selected such that the matrices $A_{\text{ii}}^d = A_{\text{ii}} - B_{\text{ii}} K_{\text{ii}} C_{\ii}^m$ and $A_{\text{hh}}^d = A_{\text{hh}} - B_{\text{hh}} K_{\text{hh}}$ are Hurwitz, then the solution
\( \epsilon(t) = [\epsilon_{bh}(t) \ \epsilon_{th}(t)]^T \) of the complete error dynamics system in (77), is GAS.

Proof: The proof is derived similarly to Theorem 1. The eigenvalues of (77) have strictly negative real parts based on the determinant property of square matrices in block triangular form.

An outline of the controller design is given in Table II. For the position tracking, the desired state variables and control inputs of the two subsystems, are the same with the ones presented in Section IV.

VI. SIMULATION RESULTS

This section provides an extensive evaluation of the proposed controller design. The controller performance is evaluated using X-Plane, a realistic and commercially available flight simulator. The experiments are conducted in the X-Plane environment for a customized Raptor 90 SE radio controlled (RC) helicopter. The parameters of the linear helicopter model of (1) are extracted using the Comprehensive Identification from Frequency Responses (CIFER) [23] software package which utilizes a frequency domain identification procedure. Frequency domain identification is regarded as a powerful method for extracting high accuracy linear dynamic models of air vehicles. This method is classified as an output-error method where the fitting error is defined between the actual flight data frequency responses and the frequency responses predicted by the model [9]. The Raptor model in X-Plane is treated as a “black box” since no a priori knowledge of the helicopter parameters is used during the identification process. The uncertainty that is injected to the model parameters through the identification procedure is very beneficial for investigating the robustness capabilities of the proposed controller. For the collection of the flight test data used by the identification process the helicopter was set to hover and specially designed excitation inputs were applied to each control command. In theory, the accuracy of the linear model derived by this process is limited in the vicinity of the hover operating condition. The identified linear model parameters for the Raptor 90 SE are given in Table III. The dashed entries indicate parameters that did not participate in the linear model. The CIFER package provides statistical confidence metrics that indicate which parameters are redundant and could be dropped off the model without effecting the identification performance. The value of these parameters was set to zero at the implementation of the control law to the Raptor.

An issue of paramount importance is the choice of the gain matrices \( K_U \) and \( K_{hh} \). As indicated earlier, these gains should be appropriately chosen such that the closed loop matrices \( A^d_U \) and \( A^d_{hh} \) are Hurwitz. Methods for the calculation of the two gain matrices are indicated in Sections IV-A and IV-B. The implementation of the aforementioned methods, will typically result in feedback gain matrices with nonzero entries. Thus, for each component of the control laws \( v_{fh}^L \) and \( v_{fh}^{hh} \), all of the feedback error terms of each subsystem will participate in the control signals.

However, in manned operations the pilot makes a distinct use of the control commands. Therefore, the pilot uses the cyclic commands for longitudinal and lateral motion, the pedal command to regulate the yaw, and the collective command to control the vertical motion of the helicopter. We would like to pass this flight intuition to the controller design. In particular we require, that each component of the feedback laws \( v_{fh}^L \) and \( v_{fh}^{hh} \) incorporates only feedback terms related to the on-axis error variables of the particular input. For example in the case of the \( v_{fh}^{ped} \) command it is preferable to use

\[
v_{fh}^{ped} = -k_f \eta h - k_{\theta} \theta - k_r r. \tag{79}
\]

In the above equation the feedback terms associated with the heave error dynamics are completely omitted. This fact makes the control law simpler and closer to the manned flight intuition. However, the aforementioned gain selection methods do not include this restrain. Algorithms which constrain certain entries of the gain matrix to zero, in the context of the optimal control, can be found in [21] and [25]–[27]. A practical approach to solve this problem is to calculate normally the feedback gains and set to zero the entries of the off-axis feedback components. It is important to check that the updated gains result in Hurwitz closed loop matrices. This approach was followed for the determination of \( K_{hh} \). The integral feedback terms of the heading-heave subsystem were not included. For the calculation of the output feedback gain matrix \( K_{hh} \), a regular eigenvalue placement algorithm was used assuming full-state feedback. Then the entries that corresponded to unmeasured states where omitted. The gain matrices of the two feedback loops which were used for conducting the experiments are shown in Table IV.

An additional comment regarding the gain selection is related to the eigenvalues of the closed-loop system. In general, for small-scale helicopters, the attitude dynamics are significant faster compared to the translational dynamics [14], [15], [28]–[31]. This physical time scaling should be accounted from the flight controller. The visualization of the two subsystems as a cascade interconnection indicates that states that belong to lower levels of each subsystem should converge faster than states of higher level. For example, the velocity dynamics of the
longitudinal-lateral subsystems should be faster than the position dynamics. Likewise, the roll and pitch angles should have faster convergence from the longitudinal and lateral velocity, respectively.

The simulation results are collected from the execution of three flight maneuvers that test the derived controller in terms of stability and tracking accuracy. The reference trajectories are specially designed in order to examine the performance of the controller in multiple operating conditions that cover a wide portion of the flight envelope. The maneuvers require aggressive flight operation, which is translated into large attitude angles and thrust magnitude. Two of the reference maneuvers involve velocity tracking while the third one requires position tracking.

The first reference maneuver is a trapezoidal velocity profile in the lateral and longitudinal directions of the inertial space. Throughout the maneuver the desired heading remains constant with a value of $\psi_h = 0$. The forward flight maneuver is composed of five parts. In the first part the helicopter is set to hover by lifting vertically 5 m from its starting point from the ground. In the second part of the maneuver, the helicopter accelerates in the longitudinal and lateral direction of the inertial space. After reaching a certain velocity the helicopter is cruising with constant speed. In the fourth part of the maneuver the helicopter decelerates until its velocity reaches zero. Then it is set to hover again. The reference velocity profile is given by

$$v_l^r(t) = \begin{cases} 
(0 
0
0)^T & \text{for } t \leq 18 \\
17 \sin \left( \frac{\pi}{34} (t - 18) \right) 
3 \sin \left( \frac{\pi}{34} (t - 18) \right) 
0 & \text{for } 18 < t \leq 35 \\
(17 
3
0)^T & \text{for } 35 < t \leq 50 \\
17 \sin \left( \frac{\pi}{40} (t - 50) \right) 
3 \sin \left( \frac{\pi}{40} (t - 50) \right) 
0 & \text{for } 50 < t \leq 70 \\
(0 
0
0)^T & \text{for } t > 70.
\end{cases}$$

The velocity responses in the inertial coordinates, versus the velocity reference trajectories with respect to time are shown in Fig. 6. The position of the helicopter in the inertial coordinates is illustrated in Fig. 7. The pitch, roll and yaw orientation angles are depicted in Fig. 8. Finally, the control inputs are given in Fig. 9.

Although that the reference trajectory requires that the helicopter executes a cruising maneuver (longitudinal velocity up to 17 m/s and lateral velocity up to 3 m/sec) a single linear controller based only on the hover linear model, was adequate. To this extent, the identification of multiple models for different operating conditions was redundant. From Fig. 8 it is apparent that the longitudinal and lateral velocities and accelerations of the helicopter are manipulated by the pitch and roll angles $\theta$, $\phi$, respectively.

The second maneuver is a more aggressive version of the previous reference trajectory. The flight task requires the helicopter to track a forward flight routine. The reference trajectory is a trapezoidal velocity profile in the $\mathbf{i}_T$ direction of the inertial frame. However, in this case the helicopter is expected to acquire higher acceleration by reaching its peak velocity in a shorter time interval. Since the longitudinal-lateral acceleration of the
helicopter has been shown to be proportional to the pitch/roll angles, a higher tilting of the fuselage is expected during the execution of this maneuver. Again, the reference heading remains constant with $\psi_r = 0$. The reference velocity trajectory profile is given by

$$v^T(t) = \begin{cases} 
(0 \ 0 \ 0)^T & \text{for } t \leq 18 \\
(22 \sin\left(\frac{\pi}{14}(t-18)\right) \ 0 \ 0)^T & \text{for } 18 < t \leq 25 \\
(0 \ 0 \ 0)^T & \text{for } 25 < t \leq 40 \\
(22 \sin\left(\frac{\pi}{40}(t-40)\right) \ 0 \ 0)^T & \text{for } 40 \leq 60 \\
(0 \ 0 \ 0)^T & \text{for } t > 60.
\end{cases}$$

The reference velocity trajectory and the actual velocity responses in the inertial coordinates are depicted in Fig. 10 with respect to time. The position of the helicopter in the inertial coordinates is shown in Fig. 11. The orientation angles during the execution of the maneuver are illustrated in Fig. 12. The generated control inputs are shown in Fig. 13. Fig. 12 indicates that due to the aggressive acceleration of the helicopter the pitch angle takes a significantly higher value compared to the previous case study. In addition, during the execution of the maneuver the helicopter reaches a peak velocity of 22 m/s. Based on extreme flight tests, the maximum possible forward velocity that the Raptor can reach is 25 m/s due to the power limitation of the main rotor.

For the third maneuver the helicopter is required to execute an “8 shaped” curved path. The heading of the helicopter remains constant throughout the execution of the maneuver. The maneuver is composed of three parts. In the first phase the helicopter lifts vertically from the starting point and is set to hover mode. In the second part of the maneuver the helicopter is expected to curve an “8 shaped” path in the longitudinal and lateral direction while its altitude remains constant. At the end of the
path the helicopter is set to hover again. The reference position trajectory is given by

\[
p^*_i(t) = \begin{cases} 
  \begin{pmatrix} 0 & 0 & -5 \end{pmatrix}^T & \text{for } t \leq 15 \\
  20 \left[ 1 - \cos \left( \frac{\pi}{20} (t - 15) \right) \right] & \text{for } 15 < t \leq 55 \\
  -14 \sin \left( \frac{\pi}{20} (t - 15) \right) & \text{for } t > 55 \\
  \begin{pmatrix} 0 & 0 & -5 \end{pmatrix}^T & \text{for } t \leq 15 \\
  \end{cases}
\]

The reference position trajectory versus the position responses in the inertial coordinates with respect to time are illustrated in Fig. 14. The position of the helicopter in the inertial coordinates is shown in Fig. 15. The orientation angles of the helicopter during the execution of the maneuver are depicted in Fig. 16. The produced control commands for this maneuver are shown in Fig. 17.

The tracking performance of the proposed controller was deemed satisfactory for all three maneuvers. Theoretically, the identified linear model of the Raptor 90 SE provides a quasi-steady dynamic description that is limited to mild flight operations (hover, cruising with low speed). However, the reference maneuvers required the transition of the helicopter into
several operating modes. In some cases, helicopter executed aggressive and high agile maneuvers that required attitude angles of up to 60°. In such operations, even the linearity assumption of the model is violated. The proposed design exhibits significant parametric robustness since a single controller, based only on the identified hover model was adequate.

A final remark worth mentioning is the following. For any helicopter controller that achieves asymptotic tracking of physically meaningful smooth reference outputs, after the elapse of possible initial transients, the helicopter states and control variables converge to a steady-state manifold which is dictated by the functional controllability of the system’s equations [32]. This fact indicates that during the execution of a reference maneuver the helicopter motion and nominal inputs are constrained. Since the proposed designed achieves asymptotic tracking, the motion of the helicopter is dictated by the desired state vector $\mathbf{x}_d$ which depends on the reference outputs and their higher derivatives. For example, based on (25) the desired pitch and roll angles are given by

$$\theta_d = \frac{1}{g} \left[ \dot{u}_r - X_{\alpha} \dot{u}_r \right]$$

$$\phi_d = \frac{1}{g} \left[ \dot{v}_r - Y_{\alpha} \dot{v}_r \right].$$

The above equations indicate that the pitch and roll angles at a steady-state condition are proportional to the reference lateral/longitudinal acceleration and velocity of the helicopter. The ability of the approximated linear model to provide the description of this steady-state manifold is attributed to the differential flatness property [14]. The knowledge of this steady-state vector can be exploited in the development of trajectory generators. For instance, from the above equation, the designer will know what attitude angles are expected during the execution of a predefined reference velocity profile.

VII. CONCLUSION

This paper presented an inertial position and heading tracking controller for small-scale unmanned helicopters. The control law is based on a generic linear model for small-scale helicopters. By disregarding the effect of the forces produced by the flapping motion of the main rotor, the approximated helicopter dynamics are differentially flat. Due to the differential flatness of the system dynamics, a desired state can be determined such that when the helicopter state vector is regulated to this desired state, the output tracking goal is achieved. The desired state is composed of the reference outputs, and it can be systematically determined using the backstepping approach. The design may be expanded such that the overall control law can be an interpolator of multiple controllers where each of them corresponds to a linearized model around different operating conditions of the helicopter. The output feedback controllers $u_{\theta, fb}$ and $u_{\phi, fb}$ are not restricted only to the proposed stabilization techniques of this paper, but they could be chosen from a wide variety of linear controller designs that exist in the literature. To this extent, the popular method of $\mathcal{H}_\infty$ may be also applied.

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers for their thorough proof reading and useful comments.

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