Exponential distributions find wide applicability in queuing theory, reliability theory and in communication theory. It is the continuous analog of the geometric distribution. The pdf of an exponential distribution is given by,

\[
    f_X(x) = \begin{cases} 
    0 & x \leq 0 \\
    \lambda e^{-\lambda x} & x > 0 
    \end{cases}
\]  
(1)

Where the parameter \( \lambda \) is the average rate at which events occur. The corresponding cdf is given by,

\[
    F_X(x) = \begin{cases} 
    0 & x \leq 0 \\
    1 - e^{-\lambda x} & x > 0 
    \end{cases}
\]  
(2)

The pdf and the cdf are shown in Fig.(1) for values of \( \lambda = 0.5, 1, 2, 3, 4 \).

The exponential distribution belongs to a class of distributions possessing memoryless (Markov) properties that we will presently show. A random variable \( X \) is called memoryless if it satisfies the condition,

\[
    P\{X > x + x_0 \mid X > x_0\} = P\{X > x\} \tag{3}
\]

The condition given by eq.(3) is equivalent to,

\[
    \frac{P\{X > x + x_0, X > x_0\}}{P\{X > x_0\}} = P\{X > x\}, \text{ Or,}
\]

\[
    P\{X > x + x_0\} = P\{X > x\} P\{X > x_0\} \tag{4}
\]

Let \( X \) be a random variable representing the time to failure of a computer with cdf given by the eq.(2). We are given that the computer has lasted \( x_0 \) hours. We want to find the conditional probability that the computer will fail at time \( x \) beyond \( x_0 \) given that it has lasted for \( x_0 \) hours. We proceed as follows to find:

\[
    P\{X \leq (x + x_0) \mid X > x_0\} = F_X(x + x_0 \mid X > x_0) = \frac{P\{X \leq (x + x_0), X > x_0\}}{P\{X \leq x_0\}} = \frac{P\{x_0 < X \leq (x + x_0)\}}{1 - P\{X \leq x_0\}} = \frac{F_X(x + x_0) - F_X(x_0)}{1 - F_X(x_0)}
\]
\[= 1 - e^{-\lambda x} e^{-\lambda x_0} \quad \text{for } x > x_0 \]

The conditional distribution \( F_X(x+x_0 \mid X > x_0) \) is the same as \( F_X(x) \). The computer forgets that it has been operating for \( x_0 \) hours and has no memory of previous hours of operation. Hence the exponential distribution is called a memoryless distribution. The corresponding conditional density \( f_X(\cdot \mid X > x_0) \) is obtained by differentiating eq. (5),

\[ f_X(x) = \lambda e^{-\lambda x}, \quad \text{for } x > x_0 \] 

**Poisson Arrival Process**

The number \( k \) of events occurring in an interval \((0, t]\) is Poisson distributed with \( p(k, \lambda) = e^{-\lambda t} \left( \frac{\lambda t}{k!} \right)^k \) and the waiting times between the occurrence one event and the next are independent and are exponentially distributed as \( e^{-\lambda t} \).

**Example 1**

A computer company leases it computers. The lifetime of one of the computers is exponentially distributed with \( \lambda = 1/50,000 \) hours. If a person leases the computer for 10,000 hours we have to find the probability that computer company will have to replace the computer before the lease runs out.

By the memoryless property derived in eq.(5) it is immaterial how long it has been leased before and hence we have,

\[ P\{\text{computer fails in } t \leq 10,000\} = 1 - e^{-10000 \lambda} = 1 - e^{-1/5} \]

and the probability that the computer will not be replaced before the lease runs out is,

\[ P\{\text{lifetime } > 10,000\} = 1 - 0.18127 = 0.81873 \]

However, if the failure time \( F_X(x) \) is not exponentially distributed then the probability of replacement has to be computed taking into account the prior use of \( x_0 \) hours of the computer. This is given by the conditional probability,

\[ P\{\text{lifetime } > x_0 + 10000 \mid \text{lifetime } > 10,000\} = \frac{1 - F_X(x_0 + 10000)}{1 - F_X(x_0)} \]

which means that we have to know \( x_0 \), the number of hours the computer was operational before the lease.

**Hazard Rate**

Given that an item has not failed up to time \( t \), the conditional probability that it will fail in the next small interval of time \( t+\Delta t \) is determined as follows:

\[ P\{t < X \leq t+\Delta t \mid X > t\} = \frac{P\{t < X \leq t+\Delta t \}}{P\{X > t\}} = \frac{P\{t < X \leq t+\Delta t\}}{P\{X > t\}} \]

where \( F_X(t) = P\{X \leq t\} \) represents the failure distribution and the quantity \( \beta(t) = \frac{f_X(t)}{1 - F_X(t)} \Delta t \) is called the *instantaneous hazard rate*. It is not a probability function as determined by the following analysis:

\[ \int_0^1 \beta(\xi) d\xi = \int_0^t \frac{f_X(\xi)}{1 - F_X(\xi)} d\xi = \int_0^t \frac{d f_X(\xi)}{1 - F_X(\xi)} = - \ln \left[1 - F_X(t)\right] \]

Since \( \int_0^1 \beta(\xi) d\xi \to \infty \) as \( t \to \infty \) we conclude that \( \beta(t) \) is not a probability function.
However, solving for $F_X(t)$, we obtain,

$$- \ln \left[ 1 - F_X(t) \right] = \int_0^t \beta(\xi) d\xi$$

$$1 - F_X(t) = \exp \left( - \int_0^t \beta(\xi) d\xi \right) = 0 \text{ as } t \rightarrow \infty$$

(8)

From eq.(7) we can obtain the cdf $F_X(t)$ given by,

$$F_X(t) = 1 - \exp \left( - \int_0^t \beta(\xi) d\xi \right)$$

(9)

Thus, $\beta(t)$ uniquely determines the failure probability distribution function $F_X(t)$. While $\beta(t)$ is not a probability density function, it is to be noted that the function $\beta(x)$ given by,

$$\beta(x) = \begin{cases} 
\frac{f_X(x)}{1 - F_X(t)} & x > t \\
0 & x \leq t
\end{cases}$$

(10)

is the conditional failure density conditioned on $X > t$ and $\beta(t)$ is the value of $\beta(x)$ at $X = t$.

Example 2
Physicians are of the opinion that the death rate of a person exercising regularly, $E$, is half that of a person who has sedentary habits, $S$. We want to compare the probabilities of survival of $E$ and $S$ given that the death rate of an exercising person $E$ is $\beta_E(t)$ and the rate for a sedentary person $S$ is $\beta_S(t)$ with $\beta_E(t) = \frac{1}{2} \beta_S(t)$.

Since the rates are conditional failure rates we will assume that both persons have lived upto the age $T_1$ and we want to compare the probabilities of their reaching the age $T_2 > T_1$. Thus we need to find,

$$P\{X_i > T_2 \mid X_i > T_1\} = \frac{P\{X_i > T_2\}}{P\{X_i > T_1\}} = \frac{1 - F_X(T_2)}{1 - F_X(T_1)}$$

where $X_i = E$ or $S$. Substituting from eq.(9) we have,

$$\frac{1 - F_X(T_2)}{1 - F_X(T_1)} = \frac{\exp \left( - \int_0^{T_2} \beta_{X_i}(\xi) d\xi \right)}{\exp \left( - \int_0^{T_1} \beta_{X_i}(\xi) d\xi \right)} = \exp \left( - \int_{T_1}^{T_2} \beta_{X_i}(\xi) d\xi \right)$$

Simplifying,

$$P\{E > T_2 \mid E > T_1\} = \exp \left( - \frac{1}{2} \int_{T_1}^{T_2} \beta_E(\xi) d\xi \right)$$

$$P\{S > T_2 \mid S > T_1\} = \exp \left( - \int_{T_1}^{T_2} \beta_S(\xi) d\xi \right)$$

Thus the probability of an exercising person who has already lived upto $T_1$ years living upto $T_2$ years is the square root of the probability of the sedentary person. If the death rate of a sedentary person, $\beta_S(t) = 0.001t$, $T_1 = 60$ years and $T_2 = 70$ years, then the following probabilities can be evaluated.

$$P\{S > 70 \mid S > 60\} = \exp \left( - \int_{60}^{70} 0.001 \xi d\xi \right) = 0.522$$

$$P\{E > 70 \mid E > 60\} = \exp \left( - \frac{1}{2} \int_{60}^{70} 0.001 \xi d\xi \right) = \sqrt{0.522} = 0.7225$$

The probability of a 60 year old sedentary person reaching 70 years is 0.522, whereas the same person exercising regularly will reach 70 years with a probability of 0.7225.