Consider a rv $X$ with pdf $f_X(x)$.

$X$ is input to a system, resulting in an output $Y$.

Output $Y$ is another RV with pdf $f_Y(y)$: Function of $f_X(x)$

Given a rule $Y = g(X)$ and $f_X(x)$: Find $f_Y(y)$.

\[
P[Y \leq y] = P[g(X) \leq y] = P[g(X) \in (-\infty, y)]
\]

\[
= P[X \in g^{-1}(-\infty, y)] = \int_{g^{-1}(-\infty,y]} f_X(x)dx
\]
Functions of Rvs: Contd.

- For every value of $X(\zeta) \in R_X$ (real line), generate a new RV $Y = g(X)$ with range $R_Y$.
- The rule $Y = g(X)$ has domain $R_X$ and range is $R_Y$.
- $Y$ is a mapping from a set of real numbers to another set of real numbers.
- Given $g(x)$ and $f_X(x)$, find the point set $C_y$ such that

$$\zeta : Y(\zeta) \leq y = \zeta : g[X(\zeta)] \leq y = \zeta : X(\zeta) \in C_y$$

Therefore: $(Y \leq y) = (X \in C_y) \rightarrow P[Y \leq y] = P[X \in C_y]$
Example 1: $Y = g(X) = x^2$

- Roots: $x = \pm \sqrt{y}$  \( y > 0 \)
- Let $X : f_X(x) : N(0, 1)$ then:

\[
P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} < X \leq \sqrt{y})
\]

\[
= \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) = \Phi(\sqrt{y}) - (1 - \Phi(\sqrt{y})
\]

\[
= 2\Phi(\sqrt{y}) - 1 = F_Y(y)
\]

To find the pdf $f_Y(y)$: $f_Y(y) = \frac{d[F_Y(y)]}{dy} = \frac{d[2\Phi(\sqrt{y}) - 1]}{dy} = 2\frac{d[\Phi(\sqrt{y})]}{dy} = \frac{1}{\sqrt{2\pi}y}e^{-\frac{y}{2}}$
Example 2

• \( y = g(x) = ax + b \) where \( a, b \in \mathbb{R} \).

• Then \( P[Y \leq y] = P[aX + b \leq y] = P[X \leq \frac{(y-b)}{a}] \quad a > 0 \)

\[
F_Y(y) = P\left[X \geq \frac{(y-b)}{a}\right] \quad a < 0
\]

\[
= 1 - P[X \leq \frac{(y-b)}{a}] \quad a < 0
\]

Therefore the pdf:

\[
f_Y(y) = \frac{dP[X \leq \frac{(y-b)}{a}]}{dy} = \frac{1}{a} f_X\left(\frac{(y-b)}{a}\right) \quad a > 0
\]

\[
f_Y(y) = \frac{1}{a} f_X\left(\frac{(y-b)}{a}\right) \quad a < 0
\]

\[
f_Y(y) = \frac{1}{|a|} f_X\left(\frac{(y-b)}{a}\right)
\]
Example 3: Y = 2X + 3. where \( f_X(x) = [u(x) - u(x - 1)] \). Show that:
\[ f_Y(y) = \frac{1}{2} f_X\left(\frac{y-3}{2}\right) \] as depicted below.
**Example 4:**

The decoder functions as follows:

- If $x(t_0) \geq 1/2$ then $y = 1$
- If $x(t_0) < 1/2$ then $y = 0$

Therefore: $Y = 0 = X < 0.5$ and $Y = 1 = X \geq 0.5$
If \( X : N(1, 1) \), then

\[
P[Y = 0] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1/2} e^{-\frac{(x-1)^2}{2}} dx = a
\]

\[
P[Y = 1] = 1 - a
\]

\[
P[Y = y] = 0 \quad \text{otherwise}
\]

Therefore \( f_Y(y) = a\delta(y) + (1 - a)\delta(y - 1) \).
Functions of Rvs: General Formula

• If \( y = g(x) \) has \( n \) real roots \( x_1, x_2, \ldots, x_n \):

  Such that at roots \( x_i \): \( y - g(x_i) = 0 \).

• Then the event \( y < Y \leq y + dy \) is the union of disjoint elementary events \( E_i \) in the Borel field of \( X \).
Consider the figure shown below where $y = g(x)$ has two roots $x_1$ and $x_2$.

- The events:

  $E_i = x_i - |dx_i| < X \leq x_i \quad if \quad g'(x_i) < 0$ and

  $E_i = x_i < X \leq x_i + |dx_i| \quad if \quad g'(x_i) > 0$
\[ P[E_i] = f_X(x_i)|dx_i| \]

\[ P[y < Y \leq y + dy] = f_Y(y)|dy| = \sum_{i=1}^{n} f_X(x_i) |dx_i| \]

\[ f_Y(y) = \sum_{i=1}^{n} f_X(x_i) \left| \frac{dx_i}{dy} \right| \]

\[ = \sum_{i=1}^{n} f_X(x_i) \left| \frac{dy}{dx_i} \right|^{-1} \]

- Since \( y = g(x) \)
  \[ \frac{dy}{dx_i} = g'(x)|_{x=x_i} \]

- \( f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{g'(x_i)} \) where \( x_i = x_i(y) \) and \( g'(x_i) \neq 0 \)

- Note: For the case where \( y - g(x) \) has no real roots, \( f_Y(y) = 0 \).
Examples: Transformations of Rvs

- \( y = \sin(x) \) and \( f_X(x) = \frac{1}{2\pi} \quad x \in (-\pi : +\pi) \)
- \( y = g(x) = x^n \) where \( x > 0 \) and \( n > 0 \)
- \( f_X(x) = 1/2 \quad x \in (0 : 2) \) and

\[
\begin{align*}
  y &= -2x + 1 \quad 0 < x < 1/2 \\
  &= 2x - 1 \quad 1/2 < x < 1
\end{align*}
\]