Given a experiment with outcomes in sample space: $\Omega$

- Probability measure applied to subsets of $\Omega$: $P[A] \geq 0$
- $P[A \cup B] = P[A] + P[B]$ if $AB = \emptyset$
- $P[A \cup B] = P[A] + P[B] - P[AB]$
- $P[A] = 1 - P[A^c]$

**Conditional Probability**

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(AB)}{P(B)}$$ (1)

**Total Probability**

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \ldots P(B|A_n)P(A_n)$$ (2)
Independence
\[ P(AB) = P(A)P(B) \]

Bayes Theorem

- \( A_i \ i = 1, \ldots, n \) : set of disjoint and exhaustive events and \( P(A_i) \neq 0 \)
- \( \bigcup_{i=1}^{n} A_i = \Omega \)
- \( A_i A_j = \emptyset \).
- \( B \) any event with \( P(B) > 0 \)

\[
P(A_j | B) = \frac{P(B | A_j)P(A_j)}{\sum_{i=1}^{n} P(B | A_i)P(A_i)} \quad (3)
\]

- \( P[A_j | B] \): Posteriori probability of \( A_j \) given \( B \)
- \( P[B|A_j] \): Apriori probability of \( B \) given \( A_j \)
- \( P[A_j] \): Causal or apriori probability of \( A_j \)
Examples

• A box contains 6 red pens and 4 black pens. Assume each pen is equally likely to be chosen. Take two pens from the box, one after the other. What is the probability both pens are black?

• There are two kids named X and Y in a room. (a) What is the probability both are boys? (b) What is the probability both are boys given that at least one of them is a boy?

• Item Z is manufactured by two companies: X and Y. X produces twice as many items produced by Y. 20% of Z supplied from X is defective. 5% of Z supplied from Y is defective. What is the probability that a randomly chosen Z is not defective. (b) If Z is defective what is probability it came from Y?
More Problems

1. A RNG generates integers from 1 to 9 (inclusive). All outcomes are equally likely and each integer is generated independently of any previous integer. Let $\Sigma = N_1 + N_2$ where $N_1$ and $N_2$ are two consecutively generated integers. (a) If it is known that $\Sigma$ is an odd number, what is the conditional probability of $\Sigma = 7$? (b) If $\Sigma > 10$, what is the probability that at least one of the integers is $> 7$? (c) If $N_1 > 8$ what is the conditional probability that $\Sigma$ will be odd?

2. An automatic breathing apparatus (B) used in anesthesia fails with probability $P_B$. A failure means death to the patient unless a monitor system $M$ detects the failure and alerts the physician. The monitor fails with probability $P_M$. The failure of these two systems are independent events. A doctor argues that if $P_M > P_B$ the installation of $M$ is useless. Considering $P_M = 0.1 = 2P_B$, determine the probability of a patient dying with and without the monitor system in place.
Permutations

- Count number of ways a particular event in an experiment can occur.
  - Enumerate all possible ways: Brute Force Approach: Tedious as number of elementary outcomes increase
  - Apply a general mathematical approach under appropriate assumptions

Consider the ordered arrangement of $n$ elements of a set $A$

- First element can be chosen $n$ different ways
- Second element can be chosen $n - 1$ different ways
- Total number of ways (or arrangements): $n(n - 1)(n - 2)...1 = n!$ (n factorial)
- Permutation of $n$ objects taken from $n$ different outcomes: $nP_n$
r-Permutation

• $r$—Permutation of set $A$: Permutation of $r$ objects taken from $n$ different elements:

• $n^P_r$ where $r \leq n$.

$$n^P_r = \frac{n(n-1)(n-2)(n-(r-1))}{(n)(n-1)...(1)} = \frac{n(n-1)(n-2)...(n-r+1)}{(n-r)(n-r-1)...1} = \frac{n!}{(n-r)!}$$

• Assumption: All $n$ objects are different and there is no replacement of the object

• If object replaced: number of ways of permuting $r$ objects: $n^r$
Combinations

- Selection/Arrangement of objects without regard to order
- Number of ways \( r \) objects selected from \( n \) with order: \( nP_r \)
- Number of ways \( r \) objects can be arranged: \( r! \)
- Combinations of \( n \) objects taken \( r \) at a time: \( nC_r \) or \( \binom{n}{r} \)

\[
\begin{align*}
nC_r &= \frac{n!}{r!(n-r)!} = \binom{n}{r} \\
\end{align*}
\]
- \( \binom{n}{r} \): Binomial Coefficients
- Other Notations: \( C^n_k \)

Example:
Let set \( A : (a, b, c) \)
Find number of ordered permutations of \( A \)
Find 2− permutation of \( A \)
Permutations of $n$ non-unique objects

- If one element is repeated $r$ times : It can be selected $r!$ ways
- Number of permutations $nP_n$ reduced to $\frac{n!}{r!}$
- In general if object $i$ repeated $n_i$ times : $n_1 + n_2 + n_k = n$
- where $k$ is the number of unique objects
- Permutations of $n$ objects: $\frac{n!}{n_1! n_2! n_3!...n_k!}$

Stirling’s Formula

- $n! \sim n^n e^{-n} \sqrt{2\pi n}$
- $\lim_{n \to \infty} \frac{n!}{n^ne^{-n}\sqrt{2\pi n}} = 1$
Examples:

• A license plate is denoted by $LLNNN$, where $L$ is a letter and $N$ is a digit. It is required that the first digit be greater than zero. Find probability of a license plate with two vowels (a,e,i,o,u) and three identical digits.

• Consider the set of elements $(1, 2, 3, 4, 5)$. What are the number of ways of forming 4 digit sequences, assuming repetitions are allowed. What is the probability of a sequence with no repeated digits?

• Consider five people denoted $(a, b, c, d, e)$ arranged in a line. What is the probability that $a$ and $b$ will be adjacent?

• Consider four pairs of people. Each pair composed of $(1M, 1F)$. Form a committee of three people. (a) How many committees possible? (b) If committee must have $(2F, 1M)$, how many possible combinations exist. (c) If the committee must consist of only one person (either M or F) per pair, how many such combinations can be formed?
• How many ways can you choose six people from a group of 7 men and 8 women with the constraint there are at least 3 W and at least 2 M in the group. (b) What is the probability of a random selection of 3 M and 3 W?
Bernoulli Trials

- Consider an experiment where outcomes: success or failure: $\Omega : (S, F)$
- Probability of a success: $p$
- Probability of a failure: $q = 1 - p$
- Conduct the experiment $n$ times: $n$ Trials
- The sample space for $n$ trials consists of $2^n$ number of possible outcomes
- Each outcome is of length $n$ and made up of a random sequence of $SSSSFSS\ldots S$
- Assuming that successive outcomes of $S$ or $F$ are independent
- Then: $Pr[SSSSFSS] = pppqpp = p^5q^1$ : Probab of 5 successes and 1 failure for $n = 6$
- But there are other combinations of $S/F$ where 5 successes and 1 failure can occur
Bernoulli Contd.

- Number of ways $k$ successes and $n - k$ failures can occur: $\binom{n}{k}$
- $\text{Prob}[k \text{ successes}; n] = \binom{n}{k} p^k q^{n-k}$ Union operation
- Note that order in which successes or failures occur is not considered here
- Bernoulli Trials: Binomial Probability Law : $b(k; n; p) = \binom{n}{k} p^k q^{n-k}$
- Probabilities: $b(k; n; p) = \binom{n}{k} p^k q^{n-k}$  \( k = 0, 1, 2..n \)
- $\sum_{k=0}^{n} b(k; n; p) = 1$
Note: Coefficient $C^n_k$ : Binomial Coefficient.

*Binomial Theorem*

$$(a + x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + ...$$

$$= + \binom{n}{n-1}ax^{n-1} + \binom{n}{n}x^n$$

Probability $b(k; n; p)$ is $(k + 1)^{th}$ coefficient of $(q + p)^n$. 
Random Variables and Probability Distributions

- Outcomes of a random experiment are not necessarily numerical values
- To facilitate mathematical analysis outcomes are mapped to a *random variable* (RV) $X$
- RV $X$ is function that maps outcome of $\Omega$ onto a number on the real line $\mathbb{R}$.
- For an outcome $\omega \in \Omega$, the RV $X : \Omega \rightarrow \mathbb{R}$
- Generate a new sample space on the $\mathbb{R}$
- Events of the original probability space replaced by events which are sets of numbers.
- Every event of $\Omega$ is a subset of $\mathbb{R}$
- Not necessary that every subset of $\mathbb{R}$ is always an event
• Subsets of Practical Interest
  - \([x = a]\)
  - \([x : a \leq x \leq b]\)
  - \([x : a < x \leq b]\)
  - \([x : a \leq x < b]\)
  - \([x : a < x < b]\)

• and their unions and intersections.

• Representations: \([a] , [a,b] , (a,b] , [a,b)\) and \((a,b)\).
SUMMARY

• Let $\Omega$ be outcomes of experiment $H$
• Elements of $\Omega$ are $\zeta$.
• To every $\zeta$ assign a real number $X(\zeta)$
• Domain of the function $X$ is $\Omega$ and its range is $R$
• Every $\zeta$ generates a specific $X(\zeta)$
• For a particular $X(\zeta)$ there may be more than one $\zeta$ that generates it.

$RV’s$ $may$ $be$ $Continuous$ $or$ $Discrete$
Cumulative Probability Distribution Functions (PDF/CDF)

RV $X$ is continuous valued:

$$F_X(x) = Pr[X \in (-\infty, x)] = Pr[X \leq x]$$

(4)

Representations: $Pr[X \leq x]$ $Pr[a < X \leq b]$.

**Properties**

- $F_X(\infty) = 1$ $F_X(-\infty) = 0$.
- $F_X(x)$ is a non-decreasing function of $x$.
- $F_X(x)$ is continuous from the right
  i.e. $F_X(x) = \lim_{\epsilon \to 0} F_X(x + \epsilon)$ $\epsilon > 0$
If point (a) or (b) are discontinuities

• $P[a \leq X \leq b] = F_X(b) - F_X(a) + P[X = a]$
• $P[a < X < b] = F_X(b) - P[X = b] - F_X(a)$
• $P[a \leq X < b] = F_X(b) - P[X = b] - F_X(a) + P[X = a]$

At continuous points $x$ $Pr[X = x] = 0$
Probability Density Function (pdf)

If \( F_X(x) \) is continuous and differentiable, then the pdf

\[
f_X(x) = \frac{dF_X(x)}{dx}
\]

(5)

The properties of the pdf:

- \( f_X(x) \geq 0 \)
- \( \int_{-\infty}^{\infty} f_X(\zeta)d\zeta = 1 = [F_X(\infty) - F_X(-\infty)] \)
- \( F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(\zeta)d\zeta \)

\[
F_X(x_2) - F_X(x_1) = \int_{-\infty}^{x_2} f_X(\zeta)d\zeta - \int_{-\infty}^{x_1} f_X(\zeta)d\zeta \\
= \int_{x_1}^{x_2} f_X(\zeta)d\zeta = P[x_1 < X \leq x_2]
\]
**Interpret \( f_X(x) \)**

- Since \( P[x < X \leq x + \Delta x] = F_X(x + \Delta x) - F_X(x) \)
- If \( F_X(x) \) is continuous in its first derivative then, as \( \Delta x \to 0 \)

\[
F_X(x + \Delta x) - F_X(x) = \int_x^{x+\Delta x} f(\zeta) d\zeta = f_X(x)\Delta x \quad (6)
\]

\[
f_X(x)\Delta x = F_X(x + \Delta x) - F_X(x)
\]
Examples of some Distributions

(univariate) Normal (Gaussian) pdf $N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{1}{2\sigma^2}(x-\mu)^2}$$ \hspace{1cm} (7)

• Mean $\mu$ and variance $\sigma^2$.

The mean and variance can be computed given the pdf as :

$$\mu = \int_{-\infty}^{\infty} x f_X(x) dx$$ \hspace{1cm} (8)

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x) dx$$ \hspace{1cm} (9)
Standard Normal Distribution : $N(0, 1)$

Given $X : N(\mu, \sigma^2)$ to determine $P(a < X \leq b)$, require integral

$$P(a < X \leq b) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b e^{-\frac{1}{2\sigma^2}[(x-\mu)^2]} \, dx$$  \hspace{1cm} (10)

Using the transformations

- $\beta = \frac{x-\mu}{\sigma}$
- $d\beta = \frac{dx}{\sigma}$

$$P(a < X \leq b) = \frac{1}{\sqrt{2\pi}} \int_{\tilde{a}}^{\tilde{b}} e^{-\frac{1}{2}x^2} \, dx$$  \hspace{1cm} (11)

where

- $\tilde{a} = \frac{(a-\mu)}{\sigma}$
- $\tilde{b} = \frac{(b-\mu)}{\sigma}$
\[ P(a < X \leq b) = \frac{1}{\sqrt{2\pi}} \int_0^b e^{-\frac{1}{2}x^2} dx - \frac{1}{\sqrt{2\pi}} \int_0^a e^{-\frac{1}{2}x^2} dx \] (12)

The error function

\[ erf(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{1}{2}t^2} dt \] (13)

is tabulated and can be calculated using upper and lower bound approximations.
Uniform distribution function

\[ f_X(x) = \frac{1}{b-a} \quad a < x < b \]  \hspace{1cm} (14)

Rayleigh Distribution

\[ f_X(x) = \frac{x}{\sigma^2} e^{\frac{x^2}{2\sigma^2}} u(x) \quad \sigma > 0 \]  \hspace{1cm} (15)

Exponential PDF

\[ f_X(x) = \frac{1}{\mu} e^{-x/\mu} u(x) \]  \hspace{1cm} (16)

where \( \mu > 0 \) and represents the average rate of occurrence of events.

Laplacian PDF

\[ f_X(x) = \frac{c}{2} e^{-c|x|} \quad c > 0 \]  \hspace{1cm} (17)