16.584: INTRODUCTION

- Theory and Tools of Probability required to analyze and design systems subject to uncertain outcomes/unpredictability/randomness.
- Such systems more generally referred to as *Experiments*.
- Need for probabilistic approaches arises for several reasons
  - Insufficient knowledge about processes contributing to the dynamics of systems that may well be deterministic.
  - The cumulative effect of a large number of processes that individually may have subdominant effects on the system.
Origins of Probability Theory

- Started in France in 1650’s in context of characterizing outcomes of gambling
- Classical principles formulated by Blaire Pascal and Pierre de Fermat
- Huygens in 1657 published first book on probability
- Bernoulli (Law of Large Numbers) and De Moivre contributors in 18th century
  - This period saw the origin of statistical theory
- Groundwork for Modern Approach to Probability:
  - Fisher and Von Mises in 1920’s: Statistical theory of probability
  - 1930’s Andrei Kolmogorov: Published axiomatic theory of probability: Basis of modern approach
Objectives

• Develop and understand general laws of probability, construct theoretical models and apply these models in various fields of engineering interest

• Consider primarily physical or statistical probability concepts where probabilities do not rely on judgements but on outcomes of the experiment

• Develop skills in handling large amounts of data/measurements generated by experiments with randomness through application of statistical reasoning and inference

• Keep an intuitive interpretation of probability to gain operational meaning in some applications.
Some Applications ...

- Design of Communication Systems and Networks subject to:
  - *Random patterns of arrivals* that utilize the resource
    * Ex: Telephone Calls Initiated at a Local Exchange Switch, Packets arriving at a network router
  - *Random holding times* of the resource
    * Ex: Telephone call durations, Packet sizes and transmission times
  - and *Uncertainty in availability and quality* of the resource
    * Ex: Wireless channels that fade, effect of noise in transmission and changes
Experiment, Sample Space, Events and Fields

• **H**: Random Experiment

• **Ω**: Sample Space consisting of all outcomes or observations of \( H \)
  
  – Set: Collection of Objects
  
  – Set of all objects under discussion: Universal Set

• **Events**: Subsets of \( Ω \): Compound or Simple Events

• Impossible (Null) Event: \( \emptyset \); Certain Event: \( Ω \)

• \( Ω \) may consist of finite, countably finite or not countably finite number of elements

• Note: If \( Ω \) consists of \( N \) elements: Total of \( 2^N \) number of subsets can exist; All subsets of \( Ω \) need not be events
Set Algebra

• \( \zeta \in \Omega : \) \( \zeta \) is an element of \( \Omega \) or belongs to set \( \Omega \). (Elementary Event)

• Union of two sets \( E_1 \) and \( E_2 \): \( E_1 \cup E_2 \) or \( E_1 + E_2 \)
  - Set of all elements in at least one of the sets \( E_1 \) or \( E_2 \).
  - \( E_1 \cup E_2 \) : \( (\zeta : \zeta \in E_1 \ or \ \zeta \in E_2 ) \)
  - Or ( \( \zeta \) lies in both \( E_1 \) and \( E_2 \) )

• \( E_1 \) is subset of \( E_2 \): \( E_1 \subset E_2 \). Then: \( E_1 \cup E_2 = E_2 \).

• Intersection of \( E_1 \) and \( E_2 \): \( E_1 \cap E_2 \) or \( E_1 E_2 \)
  - Set of all elements that are common to both \( E_1 \) and \( E_2 \)
  - \( E_1 E_2 = (\zeta : \zeta \in E_1 \ and \ \zeta \in E_2 ) \).

• Complement of \( E \): \( E^c \). Set of elements not in \( E \).
  Note: \( E \cup E^c = \Omega \).
Set Algebra Contd.

- Reduction of $E_1$ by $E_2$: $E_1 - E_2$
  - Set of elements in $E_1$ that are not in $E_2$
  - $E_1 - E_2 = E_1 \ E_2^c$
  - In general $E_1 - E_2 \neq E_2 - E_1$.

- Exclusive-Or: $E_1 \oplus E_2$: Set of elements in $E_1$ or $E_2$ but not both.
  - $E_1 \oplus E_2 = (E_1 - E_2) \cup (E_2 - E_1)$

- Disjoint sets $E_1$ and $E_2$ if $E_1 \cap E_2 = \emptyset$.
  - No elements in common. Mutually Exclusive

- $n$-partition of $E$:
  - Sequence of sets $E_i$ for $i = 1, \ldots, n$ such that:
  - $E_i \subset E$, $\bigcup_{i=1}^{N} E_i = E$ and $E_i \cap E_j = \emptyset$. 
Venn Diagrams
DeMorgan’s Laws

Given sets $E_1, \ldots, E_n$

$$\left[ \bigcup_{i=1}^{n} E_i \right]^c = \bigcap_{i=1}^{n} E_i^c$$  \hspace{1cm} (1)$$

$$\left[ \bigcap_{i=1}^{n} E_i \right]^c = \bigcup_{i=1}^{n} E_i^c$$
EVENTS

- Events of the experiment may be defined as subsets of $\Omega$.
- Consider events $A$ and $B$.
- Consider also the events $A \cup B$, $A \cap B$ and $A^c$.
  1. $A \cup B$: A or B
  2. $A \cap B$: A and B
  3. $A^c$: not A
FIELDS

• Consider $C$ as a collection of subsets of the universal set $\Omega$.

• This collection forms a field $M$ if:
  
  if $(A, B) \in M$, then
  
  $A \cup B \in M$

• if $A \in M$, then $A^c \in M$.

Any collection $M$ of subsets of $\Omega$ that satisfy the above conditions is called a field.

From the above properties it can be shown that

• $A \cap B \in M$; $\Omega \in M$; $\emptyset \in M$

$M$ is closed under finite unions and intersections.

Iff $(A_1, A_2, \ldots, A_n) \in M$ then $\cup_{i=1}^{n} A_i \in M$. 
If $\Omega$ is an infinite set define a $\sigma-$field as:

- $\emptyset \in M$, $\Omega \in M$
- if $(A_1, A_2, \ldots) \in M$ then $\bigcup_{i=1}^{\infty} A_i \in M$
- if $A \in M$, then $A^c \in M$.

A $\sigma$-field is closed under any countable set of unions, intersections and combinations.

- If $E_1, \ldots, E_n$ belong to $M$ :
  - $\bigcup_{i=1}^{\infty} E_i \in M$ and $\bigcap_{i=1}^{\infty} E_i \in M,$
SUMMARY

- Given an experiment, associate a pair $(\Omega, M)$
  - $\Omega$ is the set of all possible outcomes or elementary events
  - $M$ is the $\sigma$-field of subsets of $\Omega$
  - $M$ contains events whose occurrences are of interest
- Reference to $A$ being an event is equivalent to $A$ belongs to the $\sigma$-field in question.

Note: If $\Omega$ is not countable, such as in the case of a real line, not every subset of $\Omega$ will be assigned a probability and called an event. The smaller collection of subsets that are events consistent with the axioms forms a $\sigma$-field and on the real line this is called the Borel field of events.
Probability

- Given $[H, \Omega, F]$ 
  - where $\Omega$ has countable number of elements 
  - Every subset of $\Omega$ can be assigned a probability consistent with the axioms of Kolmogorov 

- Class of all subsets make up $F$: $\sigma$-field of events 
  - Each subset is an event 

Axiomatic Theory of Probability: $(\Omega, F, P)$

- $P$ is the probability measure. 
  - set function $P[.]$ that assigns to every event $A \in F$, a number $P[A]$ called probability
Probability Measure $P$

- $P[A] \geq 0$
- $P[\Omega] = 1$
- $P[A \cup B] = P[A] + P[B]$ if $A \cap B = \emptyset$

The above establish the following results:

- $P[\emptyset] = 0$
- $P[AB^c] = P[A] - P[AB]$ where $A \in F \ B \in F$
- $P[A] = 1 - P[A^c]$
- $P[A \cup B] = P[A] + P[B] - P[AB]$
Conditional Probabilities

• Consider two events $A$ and $B$ that are outcomes of an experiment

• Repeat experiment N times
  – $N_B$ number of occurrences of $B$
  – Consider only observations when $B$ occurred
  – Number of trials in which $A$ also occurred be $N_{(A\cap B)}$.

\[
\frac{N_{(A\cap B)}}{N_B} = \frac{N_{(A\cap B)}/N}{N_B/N} \tag{3}
\]

Interpreting these ratios as probabilities,

• $\frac{P(A\cap B)}{P(B)}$ is the probability that $A$ occurs given that $B$ has occurred.

• If $P(B) > 0$, conditional probability that $A$ occurs given that $B$ has occurred is :

\[
P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(AB)}{P(B)} \tag{4}
\]
The above probability is referred to as the probability of A given B or the probability of A conditioned (or conditional) on B. Note that $P(AB)$ is also the joint probability of events A and B.
Unconditional (Total) Probability

- Define set of mutually exclusive events $A_1, A_2, ... A_n$:
  - $\bigcup_{i=1}^{n} A_i = \Omega$
  - Note: $A_i$’s are exhaustive
- Event $B$ is defined over the probability space of the $A_i$’s.
- With $P[A_i] \neq 0$ for all $i$:

  $$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \ldots + P(B|A_n)P(A_n) \quad (5)$$

$P(B)$ is called the total or average probability of $B$. 
Independence

• Define events $A \in F$ and $B \in F$
• $P(A) > 0$ and $P(B) > 0$
• Events $A$ and $B$ said to be independent iff
  \[-P(AB) = P(A)P(B)\]
• Occurrence of event $B$ does not change probability of another event $A$.

Note: $P(AB) = P(B|A)P(A)$ in general.
For independence $P(B|A) = P(B)$ and $P(A|B) = P(A)$. 
Bayes Theorem

- \( A_i \ i = 1, \ldots, n \): set of disjoint and exhaustive events and \( P(A_i) \neq 0 \)
- \( \bigcup_{i=1}^{n} A_i = \Omega \)
- \( A_i A_j = \emptyset \).
- \( B \) any event with \( P(B) > 0 \)

\[
P(A_j | B) = \frac{P(B | A_j) P(A_j)}{\sum_{i=1}^{n} P(B | A_i) P(A_i)} \quad (6)
\]

Proof is shown by application of conditional probability to \( P(B \ A_j) \) in the numerator and total probability expression in the denominator.
Summary

- $P[A_j|B]$: Posteriori probability of $A_j$ given $B$
- $P[B|A_j]$: Apriori probability of $B$ given $A_j$
- $P[A_j]$: Causal or apriori probability of $A_j$

Apriori probabilities: Estimated from past measurements or known from experience
Aposteriori probabilities: Measured/Computed from observations
16.584: EXAMPLES

• Fair coin flipped three times. Outcomes are heads (H) or tails (T). Find probability of obtaining two heads and one tail.

• Experiment consists of drawing two balls without replacement from a box containing six balls numbered 1 to 6. Describe sample space $\Omega$. How does $\Omega$ change if the ball is replaced before the second is drawn?

• A card is selected from a deck of 52 cards. Let $A$ be event of selecting an ace and $B$ be the event of selecting a red card. There are 4 aces and 26 red cards in the deck. Are events $A$ and $B$ independent?
• Experiment is rolling two dice. Dice identified by colors red (R) and blue (B). Observations are pair of numbers shown on upper faces. Each outcome assumed equiprobable.
  
  – State sample space of problem;
  – Find probability \( R \) shows even and \( B \) shows odd numbers; Are events \( R \ even \) and \( B \ odd \) independent?
  – Find probability that sum of numbers shown by \( R \) and \( B \) is 11 and \( R \neq 5 \). Are these two events independent?
1. Consider a sample space $\Omega : \{c, d, g, p\}$. The following probabilities are given: $P[\{c, d\}] = 0.9$, $P[\{g, p\}] = 0.1$ $P[p] = 0.1$, $P[d] = 0.5$. Determine all of the subsets (16) of $\Omega$ and their probabilities.

2. In Problem 1, the information given is reduced such that, $P[\{c, d\}] = 0.9$, $P[\{g, p\}] = 0.1$ and $P[d] = 0.5$. In this case, the subset $\{g, p\}$ is to be taken as a single element, resulting in 8 subsets of $\Omega$. Find the probabilities of the field in this case.

3. You have a choice of leasing two machines $C_i, i = 1, 2$ each at 50 dollars a month or a single better machine $N_1$ at 100 dollars a month. Machines $C_1, C_2$ fail 40 percent of the time whereas machine $N_1$ fails 20 percent of the time. Assume each machine costs the same to repair.
   
   (a) If having at least one machine working is the priority which option would be best?
   
   (b) If cost of repairs is to be minimized which option would you choose?