16.202: Laplace Transforms: Review

- Finding Laplace Transforms of functions $f(t)$: $F(s) = \int_{0}^{\infty} f(t)e^{-st}dt$
- Identifying Region of Convergence
- Familiarity with LTs of common functions:
  - Exponential, Step, Delta, Cosine, Sine, Pulse etc.
- Know how to use properties of LTs and refer to the Table
  - Time Shift, Differentiation, Scaling, Multiplication by $t^n$
  - Final Value and Initial Value Theorems
- Inverse Laplace Transforms:
  - Applying Partial Fraction Expansions;
  - Finding Residues
  - Look up Inverse Functions in the Table
• Applying LTs to Circuit Elements: Resistor, Capacitor, Inductor
  – Know how to incorporate initial conditions (voltages/currents) in the LT representation of the circuit

• Application of switching elements in circuits:
  – Recognize the circuit before and after the switch is changed
  – Determine initial conditions (ICs)
  – Apply LT using the ICs to find transient and steady-state response
  – Determine the corresponding time function using Inverse Laplace Transforms
Transfer Function $H(s)$:

- Ratio of output response $Y(s)$ to input excitation $X(s)$

Assuming all initial conditions are zero

Types of Transfer Function:

- **Voltage Gain**: $H(s) = \frac{V_o(s)}{V_i(s)}$
- **Current Gain**: $H(s) = \frac{I_o(s)}{I_i(s)}$
- **Impedance**: $H(s) = \frac{V(s)}{I(s)}$
- **Admittance**: $H(s) = \frac{I(s)}{V(s)}$
• $H(s)$ characterizes the system (circuit)
  – It is independent of the specific form of input applied
• Consider Unit Impulse as Input: $x(t) = \delta(t)$
  \[ X(s) = 1 \quad \text{and} \quad H(s) = Y(s) \]
• Transfer Function is the Response to a Unit Impulse Function
• I LT of $H(s) \rightarrow h(t)$: Circuit/System Impulse Response
• $h(t) = L^{-1} [H(s)]$
• Advantage of knowing $H(s)$:
  – Given any other input $x(t) \rightarrow X(s)$, its response can be determined.
  \[ Y(s) = H(s)X(s) \quad y(t) = L^{-1}[Y(s)] \]
Example 1:

- \( H(s) = \frac{3}{s+5} \). Find \( y(t) \) if \( x(t) = u(t) \)
- Find \( X(s) \): \( X(s) = 1/s \)
- Apply \( Y(s) = X(s)H(s) = \frac{3}{s(s+5)} \)
- Find \( y(t) = L^{-1}[Y(s)] \)

\[
Y(s) = \frac{k_1}{s} + \frac{k_2}{s + 5}
\]

- \( k_1 = 3/5 \quad k_2 = -3/5 \)
- Find \( y(t) \): ILT
- \( y(t) = 3/5[u(t) - e^{-5t}u(t)] = 3/5 \left[1 - e^{-5t}\right] u(t) \)
Example 2:

1. Find transfer function $H(s) = \frac{V_o(s)}{V_i(s)}$

\[
V_o(s) = \frac{V_i(s)2/s}{1+2/s+2/s} \\
V_o(s) = \frac{2V_i(s)}{s+4} \\
H(s) = V_o(s)/V_i(s) = \frac{2}{s+4}
\]

2. Find the impulse response $h(t)$

\[
h(t) = L^{-1} \left[ \frac{2}{s+4} \right] = 2e^{-4t} u(t)
\]

3. Find $y(t)$ if input $v_i(t) = u(t)$: Step Response

\[
V_i(s) = 1/s \quad V_o(s) = \frac{2}{s(s+4)}
\]
\[ v_o(t) = \frac{1}{2}(1 - e^{-4t}) \, u(t) \]

4. Find response for \( v_i(t) = 8\cos(2t) \)

\[
V_i(s) = \frac{8s}{s^2+2^2}
\]

\[
V_o(s) = \frac{(8s)(2)}{(s^2+2^2)(s+4)}
\]

\[
V_o(s) = \frac{k_1}{s+j^2} + \frac{k_1^*}{s-j^2} + \frac{k_2}{s+4}
\]

5. Find Residues:

6. Find ILT:

7. Note: \( LT[sin(2t)] = \frac{2}{s^2+2^2} \)
Example 3

• Find impulse response of system modeled by: \( \frac{2}{dt} + y(t) = x(t) \)
• Take LT of both sides: \( 2sY(s) + Y(s) = X(s) \)
• \( 2sY(s) + Y(s) = X(s) \)
• \( H(s) = \frac{Y(s)}{X(s)} = \frac{1}{2s+1} \)
• \( H(s) = \frac{1}{2(s+1/2)} \) \( \text{ILT: } h(t) = \frac{1}{2}e^{-1/2t} u(t) \)

Example 4

Figure 3:

• Find \( H(s) = \frac{V_o(s)}{V_s(s)} \). Assume zero ICs.
Stability

- Transfer Function Determines Stability of the System
- Stability: Bounded Input produces a Bounded Output
- Requires that \( h(t) \) be bounded: \( \lim_{t \to \infty} |h(t)| : \) finite.
- Representing \( H(s) = \frac{N(s)}{D(s)} \) [Order of N(s) Order of D(s)]
- Determine poles of \( H(s) \): Roots of \( D(s) = 0 \)
  \( (s - p_1)(s - p_2) \ldots (s - p_n) = 0 \)
- Stable System: All poles must have negative real parts.
- Circuit is stable if all poles of \( H(s) \) are in left half of s-plane.
- Controlled voltage/current sources can lead to unstable circuits
Example 5

\[
\begin{align*}
    h_1(t) &= 3e^{-t}u(t) \\
    h_2(t) &= e^{-4t}u(t)
\end{align*}
\]

- Find impulse response of entire system
- Check if overall system is stable.