16.202: Frequency Response

- Frequency response of a circuit describes the behavior of the circuit with change in frequency: $\omega$
- Transfer Function: $H(s)$  
  Freq. Resp.: $s = j\omega : H(j\omega) : H(\omega)$
- $H(\omega)$ is a complex variable, a function of frequency $\omega$.
- This is an extension of the phasor analysis, where response due to a single frequency was analyzed.

\[ H(\omega) = \frac{Y(\omega)}{X(\omega)} \]

Figure 1:
• To obtain $H(\omega)$ from $H(s)$, $s = j\omega$ should be inside ROC of $H(s)$
• All poles of $H(s)$ should be in the left-half-plane (LHP)
• A stable circuit is characterized by frequency response $H(\omega)$
• $H(\omega) = |H(\omega)| \angle H(\omega)$
• Similar to $H(s)$, $H(\omega)$ can be represented as a ratio of two polynomials in $\omega$
  Complex: $H(\omega) = \frac{N(\omega)}{D(\omega)}$
  Roots of $N(\omega) = 0$ are **zeroes** of $H(\omega)$: $j\omega = z_1, z_2, \ldots$
  Roots of $D(\omega) = 0$ are **poles** of $H(\omega)$: $j\omega = p_1, p_2, \ldots$
• Convenient to use $H(s)$ to solve the problem, then replace $s = j\omega$ to determine the frequency response
Decibel Scale

- **Important**: Sketching the approximate shape of the magnitude and phase of $H(\omega)$ as a function of $\omega$.
- Decibel Scale: (dB): Measurement of Power relative to a reference level: Specify Gains
  - $G(dB) = 10 \log_{10} \left[ \frac{P}{P_{ref}} \right]$
  - Measures gain relative to $P_{ref}$
  - $P = P_{ref}$ implies 0 dB gain
  - $P < P_{ref}$ results in a negative dB value: $P = 0.5P_{ref}$: -3 dB
  - $P > P_{ref}$ Positive dB: $P = 2P_{ref}$: 3dB
In terms of voltages and currents:

\[
G_{dB} = 10\log_{10} \left( \frac{P_{out}}{P_{in}} \right)
\]

\[
= 10\log_{10} \left( \frac{V_{out}^2}{V_{in}^2} \frac{R_{out}}{R_{in}} \right) = \log_{10} \frac{V_{out}}{V_{in}} + 10\log_{10} \frac{R_{in}}{R_{out}}
\]

\[
= 20\log_{10} \frac{V_{out}}{V_{in}} \quad \text{if } R_{out} = R_{in}
\]

\[
G_{dB} = 10\log_{10} \left( \frac{P_{out}}{P_{in}} \right)
\]

\[
= 10\log_{10} \left( \frac{I_{out}^2 R_{out}}{I_{in}^2 R_{in}} \right) = 20\log_{10} \frac{I_{out}}{I_{in}} + 10\log_{10} \frac{R_{out}}{R_{in}}
\]

\[
= 20\log_{10} \frac{I_{out}}{I_{in}} \quad \text{if } R_{out} = R_{in}
\]
Bode Plots

- Semilog plot of the magnitude and phase of the frequency response
- Plot horizontal $\omega$ axis on a log scale
- Express frequency response as: $H = H(\omega) = H\angle\phi = He^{j\phi}$
  Taking natural log: $\ln H = \ln H + \ln(e^{j\phi}) = \ln H + j\phi$
- Bode Magnitude Plot: $H_{db} = 20\log_{10}\frac{H}{H(0)} = 20\log_{10}H$ (If $H(0) = 1$) vs $\omega$ (log scale)
- Bode Phase Plot: $\phi$ vs $\omega$ (log scale)
- $H = 10^{-3} \rightarrow H_{dB} = -60\text{dB}; \quad H = 10^{-2} \rightarrow H_{dB} = -40\text{dB}$
- $H = 10^{-1} \rightarrow H_{dB} = -20\text{dB}; \quad H = 0.5 \rightarrow H_{dB} = -6\text{dB}$
• $H = 0.707 \rightarrow H_{dB} = -3\text{dB}; \quad H = 1 \rightarrow H_{dB} = 0\text{dB}$

• $H = \sqrt{2} \rightarrow H_{dB} = 3\text{dB}; \quad H = 2 \rightarrow H_{dB} = 6\text{dB}$

• $H = 10^1 \rightarrow H_{dB} = 20\text{dB};$

• $H = 10^2 \rightarrow H_{dB} = 40\text{dB}$

• $H = 10^3 \rightarrow H_{dB} = 60\text{dB}$

• To generate Bode Plots, first represent $H(\omega)$ in terms of its poles and zeroes

• Apply limits as $\omega \rightarrow 0$ and $\omega \rightarrow \infty$. 