# ITERATIVE LEAST SQUARES DESIGN OF PERFECT RECONSTRUCTION QMF BANKS


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## ABSTRACT

A simple and efficient algorithm to design M-band Perfect Reconstruction (PR) Quadrature Mirror Filter (QMF) banks is presented in this paper. The originality of our approach is that the design does not rely on a computationally intensive, nonlinear optimization method but rather on an Iterative Least Squares method. The algorithm is simple to implement and rapidly converging. Design examples and a MATLAB program implementing the proposed algorithm are included.



*Keywords*—Multirate Signal Processing, PR QMF bank design, Iterative Least Squares method.


## 1. INTRODUCTION

In the past decade multirate signal processing has been a dynamic and rapidly growing field. Filter banks are used in subband coders for speech and image signals, adaptive filtering and transmultiplexing. The design of filter banks is generally performed by general purpose nonlinear optimization methods [1]. These optimization procedures are computationally very intensive, their convergences towards optimum filter banks are slow and uncertain since the cost functions have generally many local minima. As noted in [2]-[3], the starting points are critical and significant human intervention is necessary to obtain acceptable filters.

To simplify and improve the design of filter banks we have investigated ways to use an Iterative Least Squares (ILS) method. The ILS approach has been used in many filter design contexts: two dimensional minimax FIR filter design [4], log-IIR filter design [5]-[6], minimax and peak gain constrained least squares FIR filter design [7]-[8], complex Chebyshev FIR filter design [9] and allpass IIR equalizers design [10].

The use of the ILS approach to design filter banks has been proposed in [11] and [12]. However these methods are restricted to the design of 2-band QMF banks. Nayebi [13] proposed an ILS algorithm for M-band filter bank design. His design method is very general and can be used to design low delay filter banks, however the ILS method is associated with a complicated gradient based optimization method.

In this paper we present a simple and efficient ILS algorithm to design M-band PR QMF banks for $M \geq 2$. The algorithm is rapidly converging and simple to implement. Design examples and a MATLAB program implementing the proposed algorithm are presented.

## 2. PROBLEM DESCRIPTION

A typical M-band filter bank is shown in Fig. 1 where $H_k(z)$ are the analysis filters and $F_k(z)$ are the synthesis filters. The

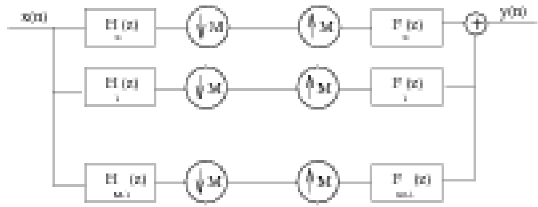


**Figure 1. Uniform M-band filter bank.**

expression of the output signal can be shown to be

$$Y(z) = \sum_{r=0}^{M-1} A_r(z) X(zW_M^r) \tag{1}$$

where

$$A_r(z) = \frac{1}{M} \sum_{k=0}^{M-1} F_k(z) H_k(zW_M^r) \tag{2}$$

and $W_M = e^{-j\frac{2\pi}{M}}$.

The perfect reconstruction property is satisfied when the reconstructed signal is a delayed version of the input signal, i.e. $y(n) = x(n - n_0)$ for some $n_0$. Hence the conditions for perfect reconstruction on the set of transfer functions $\{A_r(z)\}$ are as follows

$$A_0(z) = z^{-n_0} \tag{3}$$

$$A_r(z) = 0 \qquad 1 \leq r \leq M-1 \tag{4}$$

Three sources of distortion affecting the output signal can be distinguished identified from the expressions (1)-(4). Aliasing distortion is present when the condition (4) is not satisfied. There is phase distortion when $A_0(z)$ does not have a linear phase and magnitude distortion occurs when $|A_0(z)|$ is not strictly flat.

In this paper we address the design of cosine modulated banks [14] where the analysis and the synthesis filters are cosine modulated versions of a single linear phase lowpass FIR filter, $H(z)$, i.e. for $0 \leq k \leq M-1$

$$\begin{cases} H_k(z) = \alpha_k H(zW_{2M}^{k+.5}) + \alpha_k^* H(zW_{2M}^{-(k+.5)}) \\ F_k(z) = z^{-N} H_k(z^{-1}) \end{cases} \tag{5}$$

where $N$ is the order of the prototype filter, $\alpha_k$ is a phase factor equal to $e^{j(-1)^k \frac{\pi}{4}} W_{2M}^{(k+.5)\frac{N}{2}}$ and $\alpha_k^*$ is the complex conjugate of $\alpha_k$.The cosine modulation has three main attractive properties [1]: the aliasing between adjacent band ($A_1(z)$) and the phase distortion are entirely canceled, only one filter, i.e. the prototype filter, has to be designed and there exists a fast implementation using the discrete cosine transform. As shown in [15], the perfect reconstruction is possible when the length of the prototype filter is $N+1=2mM$, for some integer $m$, and when for $0 \leq n \leq E[(M+1)/2]-1$ and $0 \leq k \leq m-1$ the following identity holds

$$\sum_{r=0}^{2m-2k-1} h(n+rM)h(n+rM+2kM) = \frac{\delta(k)}{2M} \quad (6)$$

where $\delta(k)$ is the Kronecker symbol, i.e. $\delta(0)=1$ and $\delta(k)=0$ for $k \neq 0$ and $E[x]$ is the integer part of $x$. Note that if (6) is satisfied for $0 \leq n \leq E[(M+1)/2]-1$ , it is also satisfied for $0 \leq n \leq M-1$ thanks to the symmetry of prototype filter impulse response, i.e. $h(2mM-1-n) = \mp h(n)$ for $n = 0, ..., 2mM-1$.

As proposed in [16], when $M$ is even, the design of filter banks with small aliasing and magnitude distortions can be formulated as the following unconstrained optimization

$$min(e_s + \gamma e_d) \quad (7)$$

where $e_s$ is the stopband mean energy of the prototype filter and $e_d$ is the $L_2$ norm of the approximation error of the condition (6), i.e.

$$e_s = \frac{1}{\pi - \omega_s}\int_{\omega_s}^{\pi} |H(e^{j\omega})|^2 d\omega \quad (8)$$

$$\begin{aligned} e_d = \textstyle\sum_{k=0}^{k=m-1} \sum_{n=0}^{n=M/2-1} (\sum_{r=0}^{2m-2k-1} h(n+rM) \\ .h(n+rM+2kM) - \frac{\delta(k)}{2M})^2 \end{aligned} \quad (9)$$

and where $\omega_s$ is the stopband frequency edge and $\gamma$ is a relative weight factor associated with the minimization of $e_d$. The larger $\gamma$ is, the smaller the residual distortion in the reconstructed signal is. For $\gamma$ sufficiently large, the resulting filter banks satisfy the perfect reconstruction property.

Let us assume that the impulse response, $h(n)$ for $n = 0, 1, ..., N$, shows an even symmetry. The prototype filter frequency response can be expressed as

$$H(e^{j\omega}) = e^{-j\omega\frac{N}{2}} \sum_{n=0}^{\frac{N-1}{2}} h(\frac{N+1}{2}+n) 2cos(\omega(n+.5)) \quad (10)$$

The evaluation of the amplitude of the frequency response of the prototype filter, i.e. $A(e^{j\omega}) = e^{j\omega\frac{N}{2}}H(e^{j\omega})$, on $\Omega = \{\omega_0, ..., \omega_{L-1}\}$, a dense and uniform frequency grid of $[\omega_s, \pi]$, can be written in a matrix form as

$$[A(e^{j\omega_0}), ..., A(e^{j\omega_{L-1}})]^t = Cp \quad (11)$$

where $C$ is a $Lx\frac{N+1}{2}$ matrix containing the cosine coefficients $C_{i,j} = 2cos(\omega_{i-1}((j-1)+.5))$ and $p = [h(\frac{N+1}{2}), ..., h(N)]^t$. The discretization of (8) on the frequency grid $\Omega$ leads to the following matrix expression

$$e_s \approx \frac{1}{L}\sum_n |H(e^{j\omega_n})|^2 = \frac{1}{L}a^t a = \frac{1}{L}p^t C^t C p \quad (12)$$

## 3. PROPOSED ALGORITHM

The design of filter banks using the unconstrained optimization formulation (7) is not straightforward since the objective function is not quadratic. Complicated and computationally intensive nonlinear optimization methods can be used. To simplify and speed up the design, we propose an ILS method where the function is iteratively approximated by a quadratic quantity. The length of the prototype filter is assumed to be $N+1=2mM$. At each iteration $i$, the term $e_d$ is approximated using the prototype filter found in the $(i-1)^{\text{th}}$ iteration as follows

$$\begin{aligned} \tilde{e}_d(i) = \textstyle\frac{1}{2}\sum_{k=0}^{k=m-1} \sum_{n=0}^{n=M-1} (\sum_{r=0}^{2m-2k-1} h_i(n+rM) \\ .h_{i-1}(n+rM+2kM) - \frac{\delta(k)}{2M})^2 \end{aligned} \quad (13)$$

Note that to achieve a good approximation of (9), the summation goes from $n=0$ to $n=M-1$ and a factor $\frac{1}{2}$ is used. The quadratic term $\tilde{e}_d(i)$ can be expressed in a matrix form as follows

$$\tilde{e}_d(i) = \frac{1}{2}(Hp_i - v)^t(Hp_i - v) \quad (14)$$

where $v$ is the vector $[\frac{1}{2M}, ..., \frac{1}{2M}, 0, ..., 0]^t$ in which the first $M$ coefficients are equal to $\frac{1}{2M}$ and $H_i$ is a square matrix containing the coefficient $h_{i-1}(r)$ as follows

$$\begin{aligned} &H_i(kM+n+1, r+1) = \\ &\quad h_{i-1}(n+mM+r+2kM)\delta(r \bmod M) \\ &\quad + h_{i-1}(n+mM-1-r+2kM)\delta((r+1) \bmod M) \end{aligned} \quad (15)$$

for $0 \leq k \leq m-1$, $0 \leq n \leq M-1$, $0 \leq r \leq mM-1$ where $n \bmod l$ is the remainder of the integer division $n/r$ and $h_i(r) = 0$ for $r > 2mM-1$ and $r < 0$.

The quadratic objective function to be minimized at each iteration is as follows

$$p_i^t C^t C p_i + \frac{\gamma}{2}(H_i p_i - v)^t(H_i p_i - v) \quad (16)$$

Its minimization consists of solving the following overdetermined system of linear equations in the least squares sense

$$\begin{bmatrix} H_i \\ \sqrt{\frac{2}{L\gamma}}C \end{bmatrix} p_i = \begin{bmatrix} v \\ 0 \end{bmatrix} \quad (17)$$

The solution can be obtained using the pseudo-inverse method as follows

$$p_i = [H_i^t H_i + \frac{2}{L\gamma}C^t C]^{-1} H_i^t v \quad (18)$$

We recommend the use of the QR least squares algorithm to solve the overdetermined system (17) rather than the direct formula (18). In fact thanks to the robustness of the QR algorithm, ill conditioned situations that may occur in (18) are avoided. An implementation of the QR algorithm is provided by the MATLAB package.

To guarantee the convergence of the algorithm, we use an averaging of coefficients $p_i$ obtained from (17) or (18) as follows

$$p_i \longleftarrow \frac{p_i + p_{i-1}}{2} \quad (19)$$

Experimentally we have observed that the algorithm converges towards selective filter banks when the initial filters, $h_0(r)$, are "good" low pass filters. We have found that the stopband energy is small when $p_0$ is initialized with the coefficients of the filter that minimizes $e_s$. This filter can be obtained by setting $\gamma$ to zero at the first iteration. Moreover the stopband attenuation of the prototype filter is improved when $\gamma$ is gradually increased to its final value denoted by $\gamma_\infty$. In the implementation of the algorithm, we have $\gamma(0) = 0$, $\gamma(1) = 1$ and $\gamma(i+1) = min(K * \gamma(i), \gamma_\infty)$ for $i \geq 1$. We have

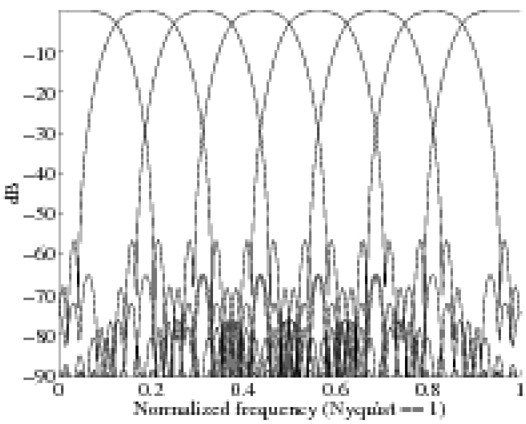


**Figure 2. Example 1: 8-band analysis filter bank.**

noticed that the slower the variation of $\gamma_i$ is, the better the stopband attenuation of the filter obtained is. The algorithm can be summarized into the following steps

1. choose $M$, $N$, $\omega_s$, $\gamma_\infty$ and $K$
2. $i = 0$, initialize $p_0(n)$ by minimizing $e_s$, set $\gamma_1 = 1$
3. $i = i + 1$,
4. build the matrix $H_i$ with $h_{i-1}(n)$ as in (15)
5. solve least squares problem using (17),
6. $\gamma(i+1) = min(K * \gamma(i), \gamma_\infty)$
7. $p_i \longleftarrow \frac{p_i + p_{i-1}}{2}$
8. go to step 3 unless $max|p_i - p_{i-1}| \leq \epsilon$, where $\epsilon$ is a prescribed small number.

## 4. DESIGN EXAMPLES

The design examples have been performed with MATLAB on a SPARC-10 workstation.

### Example 1: 8-band filter bank

Let the number of bands be $M = 8$, the length of the prototype filter be $N + 1 = 80$, the stopband frequency edge be $\omega_s = 1.5\pi/M$ and $\gamma_\infty = 10^5$. With $K = 5$ and $\epsilon = 5E^{-5}$ the algorithm converged in 12 iterations requiring 3.9 cpu seconds. The normalized analysis filters, $H_k(e^{j\omega})/\sqrt{M}$, are plotted in Fig 2. As shown in Fig. 3 the maximum value of the magnitude transfer function, $|A_0(e^{j\omega})|$, is $2E^{-5}dB$. The aliasing transfer functions, $|A_l(e^{j\omega})|$ for $l = 1, ..., 7$ are plotted in Fig. 4, their maximum level is $-116dB$.

### Example 2: 8-band PR filter bank

In this example we use the same specifications as in the example 1 except for $\gamma_\infty = 10^{22}$. With $K = 3.5$ and $\epsilon = 5E^{-5}$, the algorithm converged in 41 iterations requiring 26 cpu seconds. The normalized analysis filters, $H_k(e^{j\omega})/\sqrt{M}$, are plotted in Fig 5. As shown in Fig. 6 the maximum value of magnitude transfer function is $9E^{-14}dB$. The aliasing transfer functions are plotted in Fig. 7. The maximum level of aliasing is $-287dB$. This proves that when $\gamma_\infty$ is sufficiently large, the resulting filter bank can be considered as a PR filter bank.

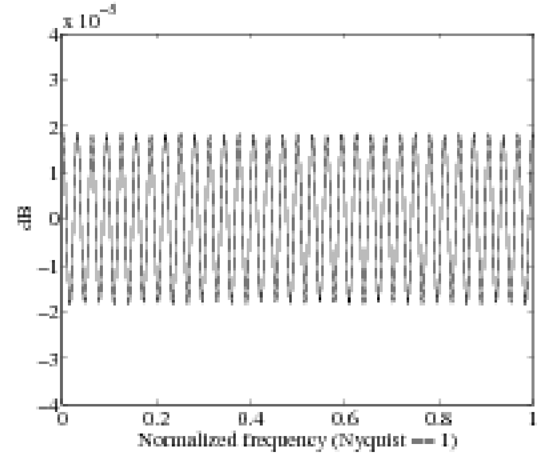


**Figure 3. Example 1: filter bank magnitude transfer function, $|A_0(e^{j\omega})|$.**

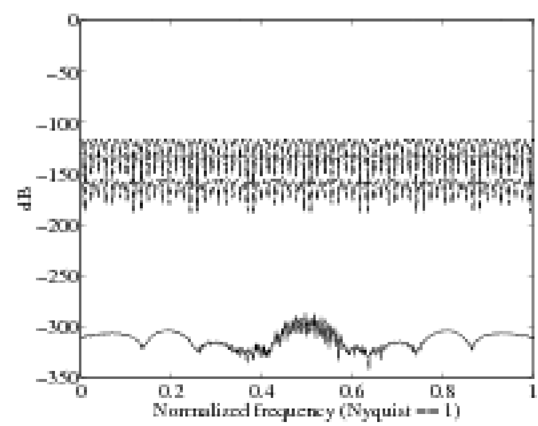


**Figure 4. Example 1: Aliasing transfer functions, $|A_r(e^{j\omega})|$ for $r = 1, ..., 7$.**

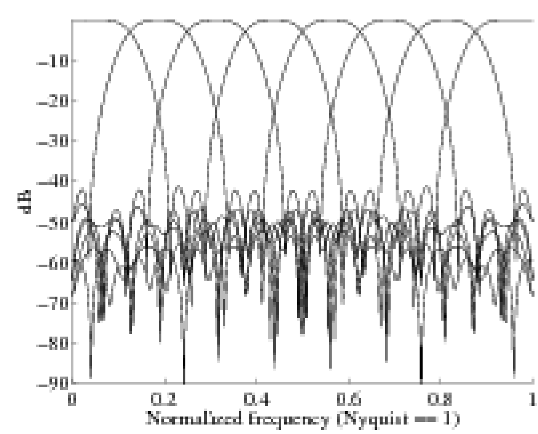


**Figure 5. Example 2: 8-band PR analysis filter bank.**

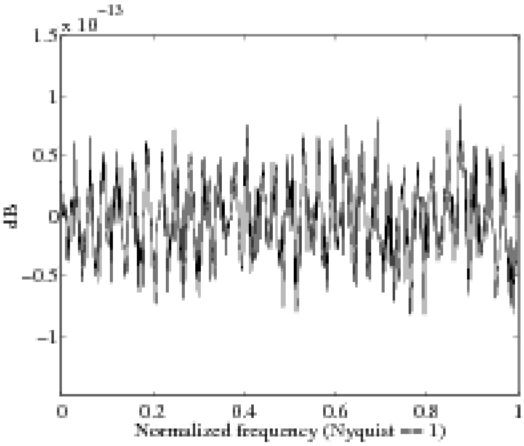


**Figure 6. Example 2: filter bank magnitude transfer function, $|A_0(e^{j\omega})|$.**

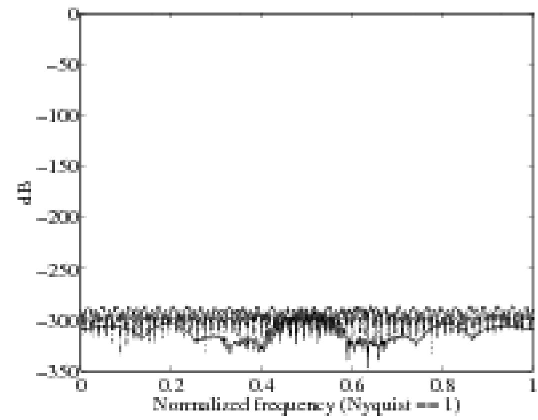


**Figure 7. Example 2: Aliasing transfer functions, $|A_r(e^{j\omega})|$ for $r = 1, \ldots, 7$.**

## 5. MATLAB PROGRAM

```
function h=PRQMF(M,Lh,fs,gammaf,K);
% PR QMF bank ILS design
% h: prototype, M: nb of bands (even),
% Lh: filter length (2Mm), fs stopband freq. >1/(2M),
% gammaf: final gamma, K: variation cate of gamma
% example 1: h=PRQMF(8,80,1.5/8,1E6,5);
% example 2: h=PRQMF(8,80,1.5/8,1E22,3.5);
% Author: Michel Rossi, University of Ottawa, 1996
epsilon=5E-6;gamma=1;to=cputime;L=4*Lh;Ad=zeros(L,1);
R=Lh/2;f=[(fs:(1-fs)/(L-1):1)]';
C=2*cos(pi*(f*([1:R]-0.5)));S=C'*C/L;
disp('i es em gamma max(delta(p))');
p=[1;-S(2:R,2:R)\S(2:R,1)];h=[flipud(p);p];
for i=1:100
 gamma=min(gamma*K,gammaf);Hi=[];pold=p;
 %Build the matrix Hi with h
 for n=0:M-1 tmp=h'.*(rem((0:Lh-1)-n,M)==0);
  for k=0:2*M:Lh-1
  if k==0 Hi=[[zeros(1,k),tmp(1:length(tmp)-k)];Hi];
  else Hi=[Hi;[zeros(1,k),tmp(1:length(tmp)-k)]];
 end;end;end;Lg=length(Hi(:,1));
 Hi=(Hi(:,R+1:Lh)+fliplr(Hi(:,1:R)));
 %Least squares solution
 v1=[ones(M,1)/2/M;zeros(Lg-M,1)];
 newp=[C/sqrt(gamma/2);Hi]\[Ad;v1];p=(p+newp)/2;
 h=[flipud(p);p];es(i)=p'*S*p;em(i)=sum(abs(Hi*p-v1).^2);
 disp([num2str(es(i)),' ',num2str(em(i))]);
 if max(abs(p-pold))<epsilon&gamma==gammaf break;
end;end;disp([num2str(cputime-to),'  cpu secs']);
```

## 6. CONCLUSIONS

In this paper we have presented an ILS algorithm for the design of Perfect Reconstruction filter banks. The algorithm is simple to implement and fast converging compared to traditional nonlinear optimization methods. Interested readers can refer to [17]-[18] for the ILS design of Pseudo QMF banks and Near Perfect Reconstruction filter banks.

## 7. ACKNOWLEDGEMENTS

The authors gratefully acknowledge support for this research under NSERC Grant No 8572 and Nortel Grant No ROTTW9596WS.

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