ITERATIVE CONSTRAINED LEAST SQUARES DESIGN OF NEAR PERFECT RECONSTRUCTION PSEUDO QMF BANKS

Michel Rossi[1] *Jin-Yun Zhang*[2] *Willem Steenaart*[3]

[1]University of Ottawa, Canada
mrossi@elg.uottawa.ca
[2]Nortel, Ottawa, Canada
jinyun@nortel.ca
[3]University of Ottawa, Canada
Life Senior Member, IEEE

ABSTRACT

This paper presents a simple and efficient method to design M-band Near Perfect Reconstruction (NPR) Pseudo Quadrature Mirror Filter (QMF) banks. This method does not rely on a conventional computationally intensive nonlinear optimization method but rather on an iterative constrained least squares algorithm. The algorithm is rapidly converging, simple to implement and flexible. Its convergence does not depend on the starting point. Moreover an iteratively calculated weighting function can be used to shape the stopband of the prototype filter and perform the minimax or the gain constrained least squares approximation. Design examples and a MATLAB program implementing the proposed algorithm are included.

Keywords—Multirate Signal Processing, NPR Pseudo QMF bank design, iterative constrained least squares algorithm.

1. INTRODUCTION

Digital filter banks are used in a number of applications such as subband coders for speech and image signals. The design of filter banks is generally performed by general purpose nonlinear optimization methods [1]. These optimization procedures are computationally very intensive, their convergences towards optimum filter banks are slow and uncertain since the cost functions have generally many local minima. As noted in [2]-[3], the starting points are critical and significant human intervention is necessary to obtain acceptable filters.

To simplify, improve and speed up the design of filter banks we have investigated ways to use an Iterative Least Squares (ILS) method. In the past decade the ILS approach has been used in many filter design contexts: two dimensional minimax FIR filter design [4], log IIR filter design [5]-[6], minimax and gain constrained least squares FIR filter design [7]-[8], complex Chebyshev FIR filter design [9] and allpass IIR equalizers design [10]. The use of the ILS approach to design filter banks has been proposed in [11] and [12]. However these methods are restricted to the design of 2-band QMF banks where the magnitude distortion is minimized but not entirely canceled. Nayebi [13] proposed an ILS algorithm for M-band filter bank design. His design method is very general and can be used to design low delay filter banks, however the ILS method is associated with a complicated gradient based optimization method.

In this paper we present a simple and efficient iterative constrained least squares algorithm to design M-band NPR Pseudo QMF banks. Thanks to the constrained formulation, the magnitude distortion is completely canceled. The algorithm is rapidly converging, simple to implement and flexible. Its convergence does not depend on the starting point. Moreover a weighting function can be iteratively calculated to design prototype filters with minimax stopbands or with gain constrained least squares stopbands.

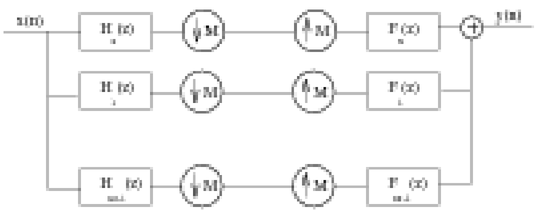

Figure 1. Uniform M-band filter bank.

2. PROBLEM DESCRIPTION

A typical M-band filter bank is shown in Fig. 1 where $H_k(z)$ are the analysis filters and $F_k(z)$ are the synthesis filters. The perfect reconstruction property is satisfied when the output signal is a delayed version of the input signal, i.e. $y(n) = x(n-n_0)$ for some number n_0. Without any design constraints on the analysis and synthesis filters, the output signal is corrupted by three kinds of distortion [1]: phase, aliasing, and magnitude distortions. In this paper we address the design of cosine modulated banks where the analysis and the synthesis filters are cosine modulated versions of a single linear phase lowpass FIR filter, $H(z)$. The cosine modulation has three main attractive properties [1]: the aliasing between adjacent bands and the phase distortion are entirely canceled, only one filter, i.e. the prototype filter, has to be designed and there exists a fast implementation using the discrete cosine transform. As shown in [14], when the residual aliasing distortion is neglected, the filter bank transfer function can be expressed as

$$T(z) = \frac{z^{-N}}{M} \sum_{k=0}^{2M-1} H(zW_{2M}^{k+.5}) H(z^{-1}W_{2M}^{-(k+.5)}) \quad (1)$$

where N is the order of the prototype filter and $W_{2M} = e^{-j\frac{\pi}{M}}$. It is easy to see that the magnitude distortion can be cancelled ($T(z) = z^{-N}$) if and only if $H(z)H(z^{-1})$ is a $2M^{th}$ band filter. The corresponding conditions on the impulse response of the prototype filter, $h(n)$ for $n = 0, ..., N$, are

$$2 \sum_{r=0}^{N-2Mk} h(r)h(2Mk+r) = \delta(k) \quad 0 \leq k \leq E[\frac{N}{2M}] \quad (2)$$

where $\delta(k)$ is the Kronecker symbol, i.e. $\delta(0) = 1$ and $\delta(k) = 0$ for $k \neq 0$ and $E[n]$ is the integer part of n.

As suggested in [14], high quality cosine modulated filter banks can be designed by minimizing the stopband mean energy of the prototype filter with the constraint that the magnitude distortion is completely canceled. These filter banks are designed by solving the following constrained minimization problem

$$min(\frac{1}{\pi - \omega_s} \int_{\omega_s}^{\pi} |H(e^{j\omega})|^2 d\omega) \text{ subject to (2)} \tag{3}$$

where ω_s is the stopband frequency edge. As pointed out in [14], the maximum level of aliasing in Pseudo QMF banks is of the order of the stopband attenuation of the prototype filter. Hence it can be kept small when the attenuation is sufficiently large for $\omega \geq \frac{\pi}{M}$.

Let us assume for the sake of simplicity that the order of the prototype filter, N, is odd and that its impulse response shows an even symmetry. The prototype filter frequency response can be expressed as

$$H(e^{j\omega}) = e^{-j\omega\frac{N}{2}} \sum_{n=0}^{\frac{N-1}{2}} h(\frac{N+1}{2} + n) 2cos(\omega(n + .5)) \tag{4}$$

The evaluation of the amplitude of the frequency response, i.e. $A(e^{j\omega}) = e^{j\omega\frac{N}{2}} H(e^{j\omega})$, on $\Omega = \{\omega_0, ..., \omega_{L-1}\}$, a dense and uniform frequency grid of $[\omega_s, \pi]$, can be written in a matrix form as

$$[A(e^{j\omega_0}), ..., A(e^{j\omega_{L-1}})]^t = Cp \tag{5}$$

where C is a Lx$\frac{N+1}{2}$ matrix containing the cosine coefficients $C_{i,j} = 2cos(\omega_{i-1}((j-1) + .5))$ and $p = [h(\frac{N+1}{2}), ..., h(N)]^t$. The evaluation of the stopband energy on Ω leads to the following expression

$$\frac{1}{\pi - \omega_s} \int_{\omega_s}^{\pi} |H(e^{j\omega})|^2 d\omega \approx \frac{1}{L} \sum_n |H(e^{j\omega_n})|^2 = \frac{1}{L} p^t C^t C p \tag{6}$$

3. PROPOSED ILS ALGORITHM

The design of NPR Pseudo QMF banks requires the solution of the constrained minimization (3). This solution can be obtained using the complicated and computationally intensive nonlinear constrained optimization method proposed in [14]. To simplify and speed up the design we propose the following Iterative Constrained Least Squares method. At each iteration i, the quadratic constraints (2) are approximated by linear constraints using the coefficients found in the previous iteration, $h_{i-1}(r)$, as follows

$$min(p_i C^t C p_i) \text{ subject to } G_i p_i = d \tag{7}$$

where p_i is the coefficient vector at the iteration i, $[h_i(\frac{N+1}{2}), ..., h_i(N)]^t$, d is the vector $[1, 0, ..., 0]^t$ and G_i is a size $(E[\frac{N}{2M}] + 1)$x$\frac{N+1}{2}$ matrix defined as follows

$$\begin{cases} G_i(1, r) = 4h_{i-1}(\frac{N+1}{2} + r - 1) & 1 \leq r \leq \frac{N+1}{2} \\ G_i(k, r) = 0 & 1 \leq r \leq M(k-1) \\ G_i(k, r) = 4h_{i-1}(\frac{N+1}{2} + r - 1 - 2M(k-1)) & \\ \quad with \quad M(k-1) + 1 \leq r \leq \frac{N+1}{2} & \end{cases} \tag{8}$$

for $2 \leq k \leq E[\frac{N}{2M}] + 1$.

The linear constrained least squares optimization problem (7) can be solved using the Lagrangian multiplier method. Let $L(p_i, \mu_i)$ be the Lagrangian corresponding to the constrained minimization (7), i.e.

$$L(p_i, \mu_i) = p_i{}^t C^t C p_i + \mu_i^t (G_i p_i - d) \tag{9}$$

where μ_i is the vector containing the Lagrangian multipliers, $[\mu_i(0), ..., \mu_i(E[\frac{N}{2M}])]^t$, associated with the linear constraints $G_i p_i = d$. The solution is obtained by setting to zero the derivative of $L(p_i, \mu_i)$ with respect to the coefficients of p_i and μ_i. This yields the following system of $\frac{N+1}{2} + E[\frac{N}{2M}] + 1$ linear equations

$$\begin{bmatrix} C^t C & G_i^t \\ G_i & 0 \end{bmatrix} \begin{bmatrix} p_i \\ \mu_i \end{bmatrix} = \begin{bmatrix} 0 \\ d \end{bmatrix} \tag{10}$$

It is easy to verify that

$$p_i = (C^t C)^{-1} G_i^t (G_i (C^t C)^{-1} G_i^t)^{-1} d \tag{11}$$

The convergence of this algorithm is not theoretically proven. However form the experimentation we have found that it does converge very fast and that the filters designed do not depend on the initial coefficients. We have also observed that the convergence speed of the algorithm depends directly on the number of constraints, i.e $E[\frac{N}{2M}] + 1$. The fewer constraints, the faster the convergence is. To speed up the convergence, the constraint in (2) with $k = 0$ can be performed by normalizing the coefficients $h_i(n)$ as follows

$$h_i(n) \longleftarrow \frac{h_i(n)}{\sqrt{2 \sum_{r=0}^{N} h_i(r)^2}} \tag{12}$$

When the number of constraints is larger than 5, we recommend the use of the QR least squares algorithm to solve (10) rather than the direct formula (11). Thanks to the robustness of the QR least squares algorithm, ill conditioned situations that may occur in (11) are avoided. An implementation of the QR algorithm is provided by the MATLAB package. The maximum value of the magnitude transfer function obtained with the proposed algorithm is typically between $10^{-12} dB$ and $10^{-14} dB$, which is comparable to that obtained with the complicated optimization method of [14]. The termination criterion can simply consist of a tolerance on the maximum variation of the coefficients of p_i. The algorithm can be summarized into the following steps

1. choose M, N and ω_s
2. $i = 0$, initialize $p_0(n)$ with random numbers
3. $i = i + 1$, build the matrix G_i with p_{i-1} using (8)
4. find p_i by solving (11) or (10)
5. go to step 3 unless $max|p_i - p_{i-1}| \leq \epsilon$, where ϵ is a prescribed small number.

An iterative re-weighted least squares method [7]-[8] can easily be coupled with the proposed algorithm to design prototype filters with minimax stopbands or gain constrained least squares stopbands. The minimax design is of interest since it guarantees the same attenuation in all rejected bands. The gain constrained least squares design offers a trade-off between the least squares and the minimax criteria. As pointed out in [15], it is possible to reduce the Chebyshev error of the least squares filter with a slight increase in the squared error. At each iteration i, the weighting function $w_i(\omega_n)$ defined on Ω is applied to the stopband of the prototype filter as follows

$$\frac{1}{L}\sum_{n} |w_i(\omega_n)H(e^{j\omega_n})|^2 = p_i^t C^t W_i^t W_i C p_i \qquad (13)$$

where $W_i = diag([w_i(\omega_0), \ldots, w_i(\omega_{L-1})])$. The constrained weighted least squares solution is obtained by replacing C by $W_i C$ in (10) or (11). When the minimax design is desired, the weights are initialized with the value 1 and updated as follows after the least squares design has converged

$$w_{i+1}(\omega_n) = w_i(\omega_n)(env(|H_i(e^{j\omega_n})|))^{\frac{\theta}{2}} \quad \text{for } \omega_n \in \Omega \quad (14)$$

where θ is a positive number typically between .1 and 2 and $env(|H_i(e^{j\omega_n})|)$ is the envelope of $|H_i(e^{j\omega_n})|$, i.e. the function which connects its maxima with straight lines. The main idea in (14) is that by using the envelope of the approximation error as a multiplicative factor, the error peak values tend to be reduced from iteration to iteration, resulting in a minimax approximation error at convergence. For the gain constrained least squares design, the weights are modified only on a subset $\hat{\Omega}$ of Ω where $env(|H_i(e^{j\omega_n})|)$ is greater than the prescribed maximum gain g_{max} as follows

$$\begin{cases} w_{i+1}(\omega_n) = w_i(\omega_n)(env(|H_i(e^{j\omega_n})|)/g_{max})^{\frac{\theta}{2}} & \text{for } \omega_n \in \hat{\Omega} \\ w_{i+1}(\omega_n) = w_i(\omega_n) & \text{for } \omega_n \in \Omega - \hat{\Omega} \end{cases} \quad (15)$$

When g_{max} is small, i.e. less than the Chebyshev norm of the corresponding minimax filter, the two weight update procedures (14) and (15) are equivalent, whereas when g_{max} is large, the weights remain unchanged and a least squares filter is designed.

4. DESIGN EXAMPLES

The design examples have been performed with MATLAB on a SPARC-10 workstation.

Example 1: 16-band NPR Pseudo QMF bank

Let the number of bands be $M = 16$, the length of the filter be $N+1 = 256$, the stopband frequency edge $\omega_s = \frac{\pi}{M}$. With $\epsilon = 1E^{-15}$ the algorithm converged in 33 iterations requiring 21 cpu seconds. The normalized analysis filter bank, $|H_k(e^{j\omega})|/\sqrt{M}$, is plotted in Fig. 2. The maximum level of aliasing is $-80dB$ which is of the same order as the stopband attenuation of the prototype filter. The overall magnitude transfer function is shown in Fig. 3. We can see that the maximum value of $|T(e^{j\omega})|$ is $2E^{-13}dB$. The normalized prototype filter, $|H(e^{j\omega})|/\sqrt{M}$, has an attenuation of $66dB$ at ω_s.

Example 2: 16-band NPR Pseudo QMF bank with a minimax prototype filter

With the same specifications as in example 1, we have designed a prototype filter with a minimax stopband. The normalized prototype filter obtained, $|H(e^{j\omega})|/\sqrt{M}$, has a minimum attenuation of $79dB$ in the stopband $[\omega_s, \pi]$. Its frequency response is plotted in Fig. 4. The maximum value of $|T(e^{j\omega})|$ and the maximum level of aliasing are the same as in example 1.

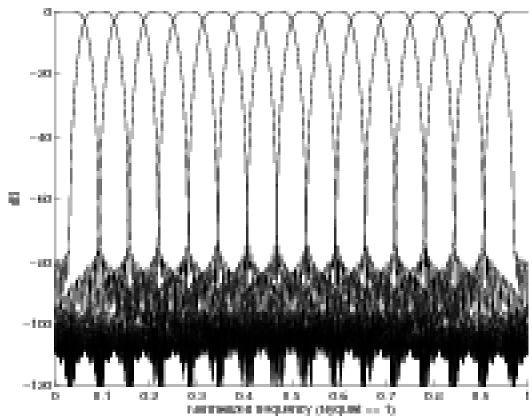

Figure 2. Example 1: 16-band NPR Pseudo QMF analysis filter bank.

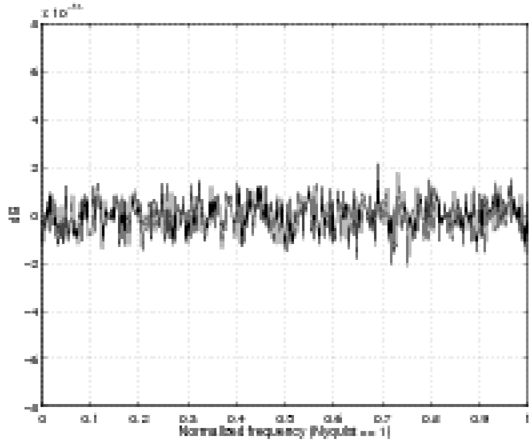

Figure 3. Example 1: filter bank magnitude transfer function, $|T(e^{j\omega})|$.

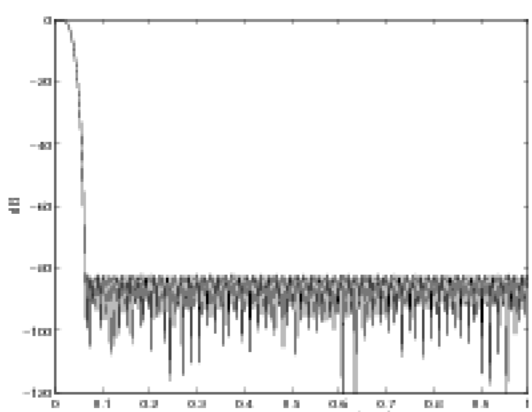

Figure 4. Example 2: minimax prototype filter of a 16-band NPR Pseudo QMF bank.

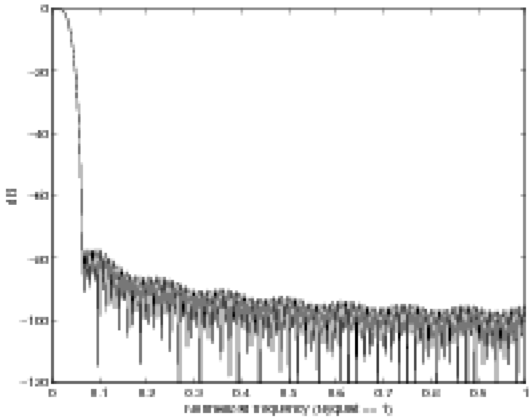

Figure 5. Example 3: gain constrained least squares prototype filter of a 16-band NPR Pseudo QMF bank.

Example 3: 16-band NPR Pseudo QMF bank with a gain constrained least squares prototype filter

With the same specifications as in example 1, we have designed a prototype filter with the least squares approximation and with the constraint $|H(e^{j\omega})|/\sqrt{M} \leq -75dB$ in the stopband $[\omega_s, \pi]$. The normalized prototype filter obtained, $|H(e^{j\omega})|/\sqrt{M}$, is shown in Fig. 5. The maximum value of $|T(e^{j\omega})|$ and the maximum level of aliasing are the same as in example 1.

5. MATLAB PROGRAM

The MATLAB function NPRQMF.m implements the proposed unweighted ILS design algorithm.

```
function h=NPRPQMF(M,Lh,f)
% NPR Pseudo QMF ILS design
% h: prototype, Lh length of h(even), M: nb of bands,
% f stopband frequency (f>1/(2M))
% example: h=NPRQMF(16,256,1/16);
% Author: Michel Rossi
% University of Ottawa, 1996
%
epsilon=1E-15;to=cputime;R=Lh/2;p=rand(R,1);
d=-sin((2*[1:R]-1)*pi*f)/pi/(1-f)/4./([1:R]-.5)+.5;
for k1=1:R for k2=k1+1:R  %Compute So
 So(k1,k2)=-[sin((k1+k2-1)*f*pi)/(k1+k2-1)+...
sin((k2-k1)*f*pi)/(k2-k1)]/2/pi/(1-f);end;end;
So=[So;zeros(1,R)];So=4*(So'+So+diag(d));v1=So(2:R,1);
S=So(2:R,2:R);disp('i es em max(delta(p))');
for i=1:40 pold=p;Go=[];
 for k=1+M:M:R  % build Go
      Go=[Go;[zeros(1,k-1),p(k-1:-1:max(1,...
      1-(R-2*(k-1))))',p(1:R-2*(k-1))']];
 end;v2=Go(:,1);K=length(v2);G=Go(:,2:R);
 % Find the Constrained L2 solution
 solp=-[[S,G'];[G,zeros(K,K)]]\[v1;v2];
 p=[1;solp(1:R-1)];p=p/sqrt(4*sum(p.^2));
 es(i)=p'*So*p;em(i)=32*sum((Go*p).^2);
 mdeltap=max(abs(pold-p));
 disp([num2str(es(i)),' ',num2str(em(i))]);
 if mdeltap<epsilon break;end;
end;disp([num2str(cputime-to),' cpu seconds']);
h=[flipud(p(1:R));p];
```

6. CONCLUSIONS

In this paper we have presented a simple algorithm for the design of NPR Pseudo QMF banks. The algorithm is fast converging and simple to implement. The convergence does not depends on the starting point. The flexibility of the ILS approach allows the use of a weighting function to shape the stopband curve of the prototype filter. Minimax and gain constrained least squares approximations can be performed. Interested readers can refer to [16]-[17] for the ILS design of Pseudo QMF banks and Perfect Reconstruction filter banks.

7. ACKNOWLEDGEMENTS

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REFERENCES

[1] P.P Vaidyanathan. *"Multirate Systems and Filter Banks,"* Englewood Cliffs, NJ: Prentice-Hall, 1992.

[2] J. D. Johnston, *"A filter family designed for use in quadrature mirror filter banks,"* ICASSP'80.

[3] R. V. Cox. *"The design of uniformly and nonuniformly spaced pseudo QMF,"* IEEE trans. Signal Proc., VOL. 34, NO. 5, pp. 1090–1096, Oct 1986.

[4] V. R Algazi, *etal*, *"Design of almost minimax FIR filters in one and two dimensions by WLS techniques,"* IEEE trans. Circuits and Syst. , VOL. 33, pp. 590-596, Jun. 1986.

[5] T. Kobayashi, S. Imai, *"Design of IIR digital filters with arbitrary log magnitude function by WLS techniques,"* IEEE trans. Sig. Process., VOL. 38, pp. 247-252, Feb 1990.

[6] M. Rossi, J. Y. Zhang. W. Steenaart, *"Two fast convergent algorithms for the design of minimax IIR and FIR filters in the Log Magnitude sense ,"* in Proc. ISCAS'96

[7] Y.C Lim. Ju-Hong Lee. C. K. Chen, Rong Huan Yang, *"A Weighted Least Squares Algorithm for Quasi-Equiripple FIR and IIR Digital Filter Design,"* IEEE trans. Sig. Process., VOL. 40, NO. 3, pp. 551–558, March 1992

[8] C. Sidney Burrus, J. A. Barreto, I. W. Selesnick, *"Iterative reweighted least-squares design of FIR filters,"* IEEE trans. Sig. Process., VOL. 42, pp. 2926-2936, Nov 1994.

[9] Ching-Yih Tseng, *"An efficient Implementation of Lawson's algorithm with application to Complex Chebyshev FIR filter Design,"* IEEE trans. Circuits and Syst. II, VOL. 42, pp. 245–260, Apr. 1995.

[10] Markus Lang. Timo I. Laasko. *"Simple and Robust Method for the Design of Allpass Filters Using Least-Squares Phase Error Criterion,"* IEEE trans. Circuits and Syst. II, VOL. 41, N0. 1, Jan. 1994

[11] Vijay K. Jain, *"Quadrature Mirror Filter Design in the Time Domain,"* IEEE trans. Sig. Process., VOL. 32. NO. 2, pp. 353–361, Apr. 1984

[12] C.-K. Chen, J.-H. Lee, *"Design of Quadrature Mirror Filters with Linear Phase in the Frequency domain,"* IEEE trans. Circuits and Syst. II, VOL. 39, N0 9, pp. 593–605 Sept 1992

[13] K. Nayebi, T. P. Barnwell, M. J. T. Smith *"Time-Domain Filter Banks Analysis: A new Design Theory,"* IEEE trans. Signal Proc., VOL. 40. NO. 6, pp. 1412–1429, June. 1992

[14] Truong Q. Nguyen. *"Near-Perfect-Reconstruction Pseudo-QMF Banks,"* IEEE trans. Sig. Process., VOL. 42, N0. 1, pp. 65–75, Jan. 1994.

[15] J. W. Adams, *"FIR digital filters with least squares stopbands subject to peak gain constraints,"* IEEE trans. Circuits and Syst., VOL. 39, N0 4 pp. 376-388, April 1991.

[16] M. Rossi, J. Y. Zhang, W. Steenaart, *"A New Algorithm for Designing Prototype Filters for M-band Pseudo QMF banks,"* EUSIPCO. Sept 1996

[17] M. Rossi, J. Y. Zhang. W. Steenaart, *"Iterative Least Squares Design of Perfect Reconstruction QMF Banks,"* CCECE'96