1. Consider the system defined by the ODE:
\[ y''' + 6y'' + 11y' + 6y = u(t) \]
where \( y \) is the output and \( u \) is the input of the system. Obtain a state-space representation of the system in terms of (a) phase variables and (b) canonical variables.

2. Given the state matrix \( A \)
\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-6 & -11 & -6 \\
\end{bmatrix}
\]
the input matrix \( B \)
\[
B = \begin{bmatrix}
0 \\
0 \\
6 \\
\end{bmatrix}
\]
the output matrix \( C \)
\[
C = \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix}
\]
and the transformation matrix
\[
P = \begin{bmatrix}
1 & 1 & 1 \\
-1 & -2 & -3 \\
1 & 4 & 9 \\
\end{bmatrix}
\]

a. Determine the transfer function where \( y \) is the output and \( u \) is the input.

b. Determine the eigenvalues of the matrix \( A \). How do they relate to the transfer function obtained in part (a).

c. If the transformation \( \tilde{x} = P\xi \) is made, determine the new system equations.

3. Consider the phase variable representation of the system matrix \( A \)
\[
A = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
-a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 \\
\end{bmatrix}
\]

Show that the transformation matrix \( P \), if \( A \) has distinct eigenvalues is

\[
P = \begin{bmatrix} e_1 | e_2 | \ldots | e_n \end{bmatrix}
\]

where each column is comprise of the eigenvalue raised to a power.

\[
e^T_m = \begin{bmatrix} \lambda_m^0 & \lambda_m^1 & \lambda_m^2 & \ldots & \lambda_m^n \end{bmatrix}
\]

In such a case \( P^{-1}AP = \Lambda \) where \( \Lambda \) is diagonal matrix comprised of the eigenvalues of matrix \( A \).

4. Consider the frictionless pendulum moving on a flat surface. The objective is to hold the pendulum vertical for small displacements in angle.

\[
\frac{4}{3} ml^2 \theta'' + mly'' + mgl\theta = 0
\]

\[
ml\theta'' + (M + m)y'' = u
\]

where \( M = 3 \text{ kg}, \ l = 2 \text{ meters} \ , m = 1 \text{ kg} \ \text{and} \ g = 9.8 \text{ meters/sec}^2 \). If \( x_1 = \theta \), \( x_2 = \theta' \), \( x_3 = y \) and \( x_4 = y' \).
a. Write the state equations $\dot{x} = Ax + Bu$ Note the system can be reduced to two independent state-variables $x_1$ and $x_2$ by substitution for $y$ in the first equation.

b. Find the eigenvalues of $A$.

c. Find the eigenvectors of $A$.

d. Find the similarity transformation, $P$ such that $P^{-1}AP = \Lambda$.

e. Determine the controllability of the system.

f. Determine the matrix $C$ required for observability if one observation is made $y = Cx$. 