d. The phase angle θ between p_r and u_r may be found from Eq. (2.64).

$$\theta = 90° - \tan^{-1} kr = 90° - 79.6° = 10.4°$$

e. The energy density is given by Eq. (2.84).

$$D_{avg} = \frac{|p_r|^2}{\rho_0 c^2}\left(1 + \frac{1}{2k^2r^2}\right) = \frac{360}{1.4 \times 10^5}\left(1 + \frac{1}{2 \times 29.8}\right)$$
$$\doteq 2.62 \times 10^{-3} \text{ watt-sec/m}^3$$

f. The sound pressure level is found from Eq. (1.18).

$$\text{SPL} = 20 \log_{10} \frac{18.97}{2 \times 10^{-5}}$$
$$= 119.5 \text{ db } re\ 2 \times 10^{-5} \text{ newton/m}^2\ (re\ 2 \times 10^{-4} \text{ microbar})$$

This sound pressure level is about 15 db higher than the highest level that is measured at 25 ft above a full symphony orchestra. In other words, 1 watt of acoustic power creates a very high sound pressure level at 1 ft from the source.

CHAPTER 3

ELECTRO-MECHANO-ACOUSTICAL CIRCUITS

PART VI *Mechanical Circuits*

3.1. Introduction. The subject of electro-mechano-acoustics (sometimes called dynamical analogies) is the application of electrical-circuit theory to the solution of mechanical and acoustical problems. In classical mechanics, vibrational phenomena are represented entirely by differential equations. This situation existed also early in the history of telephony and radio. As telephone and radio communication developed, it became obvious that a schematic representation of the elements and their interconnections was valuable. These schematic diagrams made it possible for engineers to visualize the performance of a circuit without laboriously solving its equations. The performance of radio and television systems can be studied from a single sheet of paper when such schematic diagrams are used. Such a study would have been hopelessly difficult if only the equations of the system were available.

There is another important advantage of a schematic diagram besides its usefulness in visualizing the system. Often one has a piece of equipment for which he desires the differential equations. The schematic diagram may then be drawn from visual inspection of the equipment. Following this, the differential equations may be formed directly from the schematic diagrams. Most engineers are trained to follow this procedure rather than to attempt to formulate the differential equations directly.

Schematic diagrams have their simplest applications in circuits that contain lumped elements, *i.e.*, where the only independent variable is time. In distributed systems, which are common in acoustics, there may be as many as three space variables and a time variable. Here, a schematic diagram becomes more complicated to visualize than the differential equations, and the classical theory comes into its own again. There are many problems in acoustics, however, in which the elements are lumped and the schematic diagram may be used to good advantage.

Four principal requirements are fulfilled by the methods used in this text to establish schematic representations for acoustic and mechanical devices. They are:

1. The methods must permit the formation of schematic diagrams from visual inspection of devices.

2. They must be capable of such manipulation as will make possible the combination of electrical, mechanical, and acoustical elements into one schematic diagram.

3. They must preserve the identity of each element in combined circuits so that one can recognize immediately a force, voltage, mass, inductance, and so on.

4. They must use the familiar symbols and the rules of manipulation for electrical circuits.

Several methods that have been devised fulfill one or two of the above four requirements, but not all four. A purpose of this chapter is to present a new method for handling combined electrical, mechanical, and acoustic systems. It incorporates the good features of previous theories and also fulfills the above four requirements. The symbols used conform with those of earlier texts wherever possible.[1-5]

3.2. Physical and Mathematical Meanings of Circuit Elements. The circuit elements we shall use in forming a schematic diagram are those of electrical-circuit theory. These elements and their mathematical meaning are tabulated in Table 3.1 and should be learned at this time. There are generators of two types. There are four types of circuit elements: resistance, capacitance, inductance, and transformation. There are three generic quantities: (*a*) the drop across the circuit element; (*b*) the flow through the circuit element; and (*c*) the magnitude of the circuit element.†

Attention should be paid to the fact that the quantity a is not restricted to voltage e, nor b to electrical current i. In some problems a will represent force f, or velocity u, or pressure p, or volume velocity U. In those cases b will represent, respectively, velocity u, or force f, or volume velocity U, or pressure p. Similarly, the quantity c might be any appropriate quantity such as mass, compliance, inductance, resistance, etc. The physical meaning of the circuit elements c depends on the way in which the quantities a and b are chosen, with the restriction that ab has the dimension of power in all cases. The complete array of alternatives is shown in Table 3.2.

[1] B. Gehlshoj, "Electromechanical and Electroacoustical Analogies," Academy of Technical Sciences, Copenhagen, 1947.

[2] F. A. Firestone, A New Analogy between Mechanical and Electrical Systems, *J. Acoust. Soc. Amer.*, **4**: 249–267 (1933); The Mobility Method of Computing the Vibrations of Linear Mechanical and Acoustical Systems: Mechanical-electrical Analogies, *J. Appl. Phys.*, **9**: 373–387 (1938).

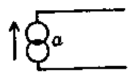
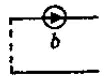
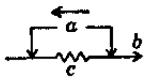
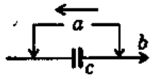
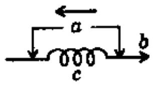
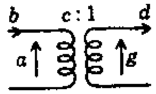
[3] H. F. Olson, "Dynamical Analogies," D. Van Nostrand Company, Inc., New York, 1943.

[4] W. P. Mason, Electrical and Mechanical Analogies, *Bell System Tech. J.*, **20**: 405–414 (1941).

[5] A. Bloch, Electro-mechanical Analogies and Their Use for the Analysis of Mechanical and Electro-mechanical Systems, *J. Inst. Elec. Eng.*, **92**: 157–169 (1945).

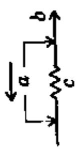
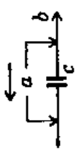
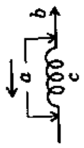
† Among the four circuit elements, the first three are two-poles. This list is exhaustive. The transformation element is a four-pole. There are other lossless four-poles which one might have chosen in addition, *e.g.*, the ideal gyrator.

TABLE 3.1. Mathematical and Physical Significance of Symbols

Symbol	Name	Meaning	
		Transient	Steady-state
	Constant-drop generator	The quantity a is independent of what is connected to the generator. The arrow points to the positive terminal of the generator	
	Constant-flow generator	The quantity b is independent of what is connected to the generator. The arrow points in the direction of positive flow	
	Resistance-type element	$a = bc$	$a = bc$
	Capacitance-type element	$a = \frac{1}{c}\int b\,dt$	$a = \frac{b}{j\omega c}$
	Inductance-type element	$a = c\frac{db}{dt}$	$a = j\omega cb$
	Transformation-type element	$a = cg$ $cb = d$ $\frac{a}{b} = c^2\frac{g}{d}$	$a = cg$ $cb = d$ $\frac{a}{b} = c^2\frac{g}{d}$

An important idea to fix in your mind is that the *mathematical operations associated with a given symbol are invariant.* If the element is of the inductance type, for example, the drop a across it is equal to the time derivative of the flow b through it multiplied by its size c. Note that this rule is not always followed in electrical-circuit theory because there conductance and resistance are often indiscriminately written beside the symbol for a resistance-type element. The invariant operations to be associated with each symbol are shown in columns 3 and 4 of Table 3.1.

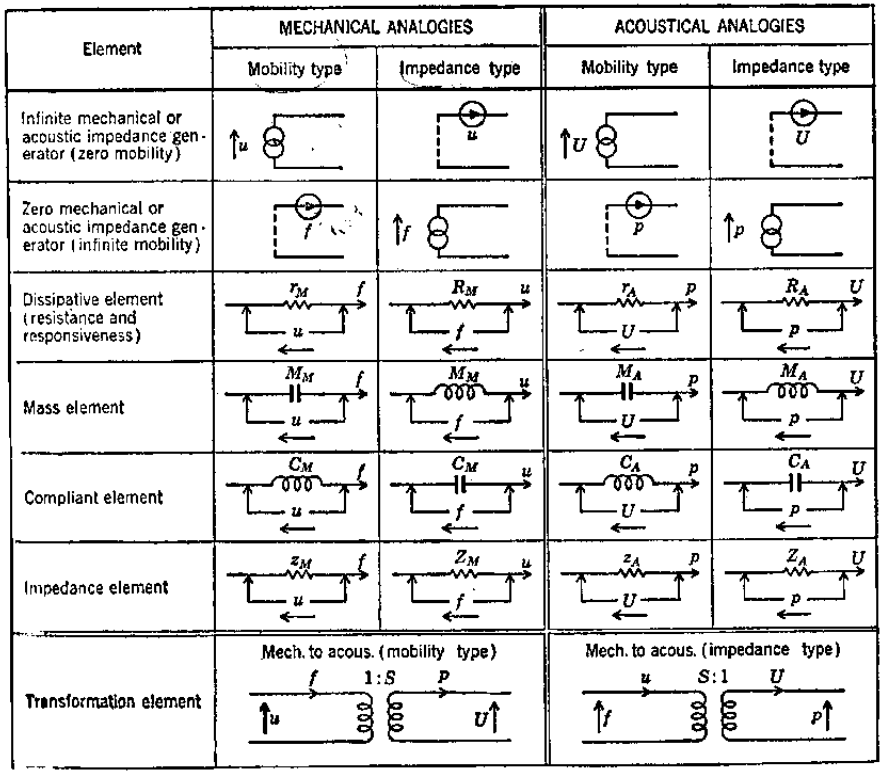
TABLE 3.2. Values for a, b, and c in Electrical, Mechanical, and Acoustical Circuits

Element	Electrical	Mechanical: Mobility analogy†	Mechanical: Impedance analogy	Acoustical: Impedance analogy†	Acoustical: Mobility analogy
a	e	u	f	p	U
b	i	f	u	U	p
	$c = R_E$	$c = \frac{1}{R_M} = r_M$	$c = R_M$	$c = R_A$	$c = \frac{1}{R_A} = r_A$
	$c = C_E$	$c = M_M$	$c = C_M$	$c = C_A$	$c = M_A$
	$c = L$	$c = C_M$	$c = M_M$	$c = M_A$	$c = C_A$
	$c = Z_E = \frac{e}{i}$	$c = z_M = \frac{u}{f} = \frac{1}{Z_M}$	$c = Z_M = \frac{f}{u} = \frac{1}{z_M}$	$c = Z_A = \frac{p}{U} = \frac{1}{z_A}$	$c = z_A = \frac{U}{p} = \frac{1}{Z_A}$

† Preferred analogies.

3.3. Mechanical Circuits. Mechanical-circuit elements need not always be represented by electrical symbols. Since one frequently draws a mechanical circuit directly from inspection of the mechanical device, more obvious forms of mechanical elements are sometimes useful, at least until the student is thoroughly familiar with the analogous circuit. We shall accordingly devise a set of "mechanical" elements to be used as an introduction to the elements of Table 3.1.

TABLE 3.3. Conversion from Mobility-type Analogy to Impedance-type Analogy, or Vice Versa



In electrical circuits, a voltage measurement is made by attaching the leads from a voltmeter *across* the two terminals of the element. Voltage is a quantity that we can measure without breaking into the circuit. To measure electric current, however, we must break into the circuit because this quantity acts *through* the element. In mechanical devices, on the other hand, we can measure the velocity (or the displacement) without disturbing the machine by using a capacitive or inertially operated vibration pickup to determine the quantity at any point on the machine. It is not velocity but force that is analogous to electric current. Force cannot be measured unless one breaks into the device.

It becomes apparent then that if a mechanical element is strictly analogous to an electrical element it must have a velocity difference appearing between (or across) its two terminals and a force acting through it. Analogously, also, the product of the rms force f in newtons and the in-phase component of the rms velocity u in meters per second is the power in watts. We shall call this type of analogy, in which a velocity corresponds to a voltage and a force to a current, the *mobility-type analogy*. It is also known as the "inverse" analogy.

Many texts teach in addition a "direct" analogy. It is the opposite of the mobility analogy in that force is made to correspond to voltage and velocity to current. In this text we shall call this kind of analogy an *impedance-type analogy*. To familiarize the student with both concepts, all examples will be given here both in mobility-type and impedance-type analogies.

Mechanical Impedance Z_M, and Mechanical Mobility z_M. The mechanical impedance is the complex ratio of force to velocity at a given point in a mechanical device. We commonly use the symbol Z_M for mechanical impedance, where the subscript M stands for "mechanical." The units are newton-seconds per meter, or mks mechanical ohms.

The mechanical mobility is the inverse of the mechanical impedance. It is the complex ratio of velocity to force at a given point in a mechanical device. We commonly use the symbol z_M for mechanical mobility. The units are meters per second per newton, or mks mechanical mohms.†

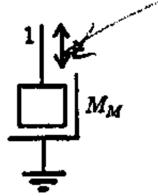


FIG. 3.1. Mechanical symbol for a mass.

Mass M_M. Mass is that physical quantity which when acted on by a force is accelerated in direct proportion to that force. The unit is the kilogram. At first sight, mass appears to be a one-terminal quantity because only one connection is needed to set it in motion. However, the force acting on a mass and the resultant acceleration are reckoned with respect to the earth (inertial frame) so that in reality the second terminal of mass is the earth.

The mechanical symbol used to represent mass is shown in Fig. 3.1. The upper end of the mass moves with a velocity u with respect to the ground. The ⊥-shaped configuration represents the "second" terminal of the mass and has zero velocity. The force can be measured by a suitable device inserted between the point 1 and the next element or generator connecting to it.

Mass M_M obeys Newton's second law that

$$f(t) = M_M \frac{du(t)}{dt} \tag{3.1}$$

† The word "mohm" stands for mobility ohm. The units are meters per second per newton.

where $f(t)$ is the instantaneous force in newtons, M_M is the mass in kilograms, and $u(t)$ is the instantaneous velocity in meters per second.

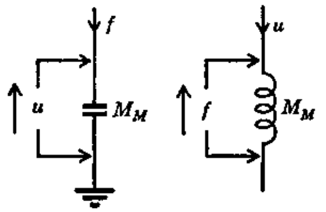
In the steady state [see Eqs. (2.33) to (2.35)], with an angular frequency ω equal to 2π times the frequency of vibration, we have the special case of Newton's second law,

$$f = j\omega M_M u \tag{3.2}$$

where $j = \sqrt{-1}$ as usual and f and u are rms complex quantities.

The mobility-type analogous symbol that we use as a replacement for the mechanical symbol in our circuits is a capacitance type. It is shown in Fig. 3.2*a*. The mathematical operation invariant for this symbol is found from Table 3.1. In the steady state we have

$$a = \frac{b}{j\omega c} \qquad \text{or} \qquad u = \frac{f}{j\omega M_M} \tag{3.3}$$



Mobility-type (*a*) Impedance-type (*b*)

FIG. 3.2. (*a*) Mobility-type and (*b*) impedance-type symbols for a mass.

This equation is seen to satisfy the physical law given in Eq. (3.2). Note the similarity in appearance of the mechanical and analogous symbols in Figs. 3.1 and 3.2*a*. In electrical circuits the time integral of the current through a capacitor is charge. The analogous quantity here is the time integral of force, which is momentum.

The impedance-type analogous symbol for a mass is an inductance. It is shown in Fig. 3.2*b*. The invariant operation for steady state is $a = j\omega cb$ or $f = j\omega M_M u$. It also satisfies Eq. (3.2). Note, however, that in this analogy one side of the mass element is not necessarily grounded; this often leads to confusion. In electrical circuits the time integral of the voltage across an inductance is flux-turns. The analogous quantity here is momentum.

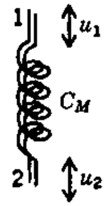


FIG. 3.3. Mechanical symbol for a mechanical compliance.

Mechanical Compliance C_M. A physical structure is said to be a mechanical compliance C_M if, when it is acted on by a force, it is displaced in direct proportion to the force. The unit is the meter per newton. Compliant elements usually have two apparent terminals.

The mechanical symbol used to represent a mechanical compliance is a spring. It is shown in Fig. 3.3. The upper end of the element moves with a velocity u_1 and the lower end with a velocity u_2. The force required to produce the difference between the velocities u_1 and u_2 may be measured by breaking into the machine at either point 1 or point 2. Just as the same current would be measured at either end of an element in an electrical circuit, so the same force will be found here at either end of the compliant element.

Mechanical compliance C_M obeys the following physical law,

$$a = \frac{1}{c}\int b\,dt \qquad \text{or} \qquad f(t) = \frac{1}{C_M}\int u(t)\,dt \tag{3.4}$$

where C_M is the mechanical compliance in meters per newton and $u(t)$ is the instantaneous velocity in meters per second equal to $u_1 - u_2$, the difference in velocity of the two ends.

In the steady state, with an angular frequency ω equal to 2π times the frequency of vibration, we have,

$$f = \frac{u}{j\omega C_M} \tag{3.5}$$

where f and u are taken to be rms complex quantities.

The mobility-type analogous symbol used as a replacement for the mechanical symbol in our circuits is an inductance. It is shown in Fig. 3.4*a*. The invariant mathematical operation that this symbol represents is given in Table 3.1. In the steady state we have

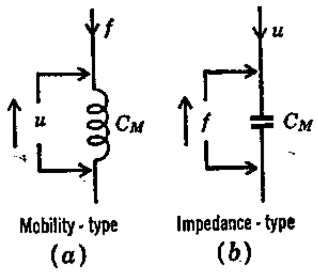


FIG. 3.4. (*a*) Mobility-type and (*b*) impedance-type symbols for a mechanical compliance.

$$u = j\omega C_M f \tag{3.6}$$

In electrical circuits the time integral of the voltage across an inductance is flux-turns. The analogous quantity here is the time integral of velocity, which is displacement.

This equation satisfies the physical law given in Eq. (3.5). Note the similarity in appearance of the mechanical and analogous symbols in Figs. 3.3 and 3.4*a*.

The impedance-type analogous symbol for a mechanical compliance is a capacitance. It is shown in Fig. 3.4*b*. The invariant operation for steady state is $a = b/j\omega c$, or $f = u/j\omega C_M$. It also satisfies Eq. (3.5). In electrical circuits the time integral of the current through a capacitor is the charge. The analogous quantity here is the displacement.

Mechanical Resistance R_M, and *Mechanical Responsiveness* r_M. A physical structure is said to be a mechanical resistance R_M if, when it is acted on by a force, it moves with a velocity directly proportional to the force. The unit is the mks mechanical ohm.

We also define here a quantity r_M, the mechanical *responsiveness*, that is the reciprocal of R_M. The unit of responsiveness is the mks mechanical mohm.

The above representation for mechanical resistance is usually limited to viscous resistance. Frictional resistance is excluded because, for it, the ratio of force to velocity is not a constant. Both terminals of resistive elements can usually be located by visual inspection.

The mechanical element used to represent viscous resistance is the fluid dashpot shown schematically in Fig. 3.5. The upper end of the element moves with a velocity u_1 and the lower with a velocity u_2. The force required to produce the difference between the two velocities u_1 and u_2 may be measured by breaking into the machine at either point 1 or point 2.

Mechanical resistance R_M obeys the following physical law,

$$f = R_M u = \frac{1}{r_M} u \tag{3.7}$$

where f is the force in newtons, u is the difference between the velocities u_1 and u_2 of the two ends, R_M is the mechanical resistance in mechanical ohms, *i.e.*, newtons/(meter per second), and r_M is the mechanical responsiveness in mks mechanical mohms, *i.e.*, meters per second per newton.

The mobility-type analogous symbol used to replace the mechanical symbol in our circuits is a resistance. It is shown in Fig. 3.6*a*. The invariant mathematical operation that this symbol represents is given in Table 3.1. In either the steady or transient state we have

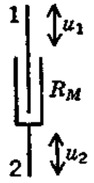


FIG. 3.5. Mechanical symbol for mechanical (viscous) resistance.

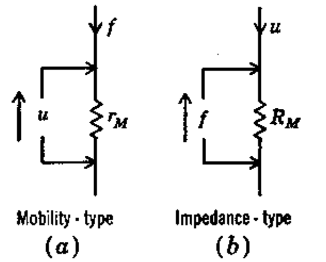


FIG. 3.6. (*a*) Mobility-type and (*b*) impedance-type symbols for a mechanical resistance.

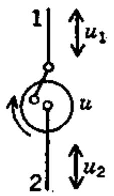


FIG. 3.7. Mechanical symbol for a constant-velocity generator.

$$u = r_M f = \frac{1}{R_M} f \tag{3.8}$$

In the steady state u and f are taken to be rms complex quantities. This equation satisfies the physical law given in Eq. (3.7).

The impedance-type analogous symbol for a mechanical resistance is shown in Fig. 3.6*b*. It also satisfies Eq. (3.7).

Mechanical Generators. The mechanical generators considered will be one of two types, constant-velocity or constant-force. A *constant-velocity generator* is represented as a very strong motor attached to a shuttle mechanism in the manner shown in Fig. 3.7. The opposite ends of the generator have velocities u_1 and u_2. One of these velocities, either u_1 or u_2, is determined by factors external to the generator. The difference between the velocities u_1 and u_2, however, is a velocity u that is independent of the external load connected to the generator.

The symbols that we used in the two analogies to replace the mechanical symbol for a constant-velocity generator are shown in Fig. 3.8. The invariant mathematical operations that these symbols represent are also given in Table 3.1. The tips of the arrows point to the "positive" terminals of the generators. The double circles in Fig. 3.8*a* indicate that the internal mobility of the generator is zero. The dashed line in Fig. 3.8*b* indicates that the internal impedance of the generator is infinite.

A *constant-force generator* is represented here by an electromagnetic transducer (*e.g.*, a moving-coil loudspeaker) in the primary of which an electric current of constant amplitude is maintained. Such a generator produces a force equal to the product of the current i, the flux density B, and the effective length of the wire l cutting the flux ($f = Bli$). This device is shown schematically in Fig. 3.9. The opposite ends of the

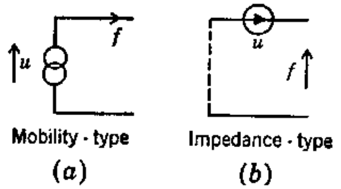


FIG. 3.8. (*a*) Mobility-type and (*b*) impedance-type symbols for a constant-velocity generator.

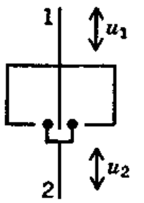


FIG. 3.9. Mechanical symbol for a constant-force generator.

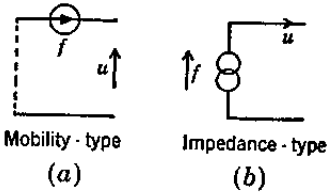


FIG. 3.10. (*a*) Mobility-type and (*b*) impedance-type symbols for a constant-force generator.

generator have velocities u_1 and u_2 that are determined by factors external to the generator. The force that the generator produces and that may be measured by breaking into the device at either point 1 or point 2 is a constant force, independent of what is connected to the generator.

The symbols used in the two analogies to replace the mechanical symbol for a constant-force generator are given in Fig. 3.10. The invariant mathematical operations that these symbols represent are also given in Table 3.1. The arrows point in the direction of positive flow. Here, the dashed line indicates infinite mobility, and the double circles indicate zero impedance.

Levers. SIMPLE LEVER. It is apparent that the lever is a device closely analogous to a transformer. The lever in its simplest form consists of a weightless bar resting on an immovable fulcrum, so arranged that a downward force on one end causes an upward force on the other end (see Fig. 3.11). From elementary physics we may write the equation of balance of moments around the fulcrum,

$$f_1 l_1 = f_2 l_2$$

or, if not balanced, assuming small displacements,

$$u_1 l_2 = u_2 l_1 \tag{3.9}$$

Also,

$$\begin{aligned} z_{M1} &= \frac{u_1}{f_1} = \left(\frac{l_1}{l_2}\right)^2 z_{M2} \\ Z_{M1} &= \frac{f_1}{u_1} = \left(\frac{l_2}{l_1}\right)^2 Z_{M2} \end{aligned} \tag{3.10}$$

The above equations may be represented by the ideal transformers of Fig. 3.12, having a transformation ratio of $\left(\frac{l_1}{l_2}\right):1$ for the mobility type and $\left(\frac{l_2}{l_1}\right):1$ for the impedance type.

FLOATING LEVER. As an example of a simple floating lever, consider a weightless bar resting on a fulcrum that yields under force. The bar is

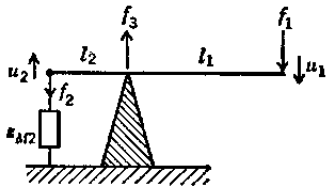


FIG. 3.11. Simple lever.

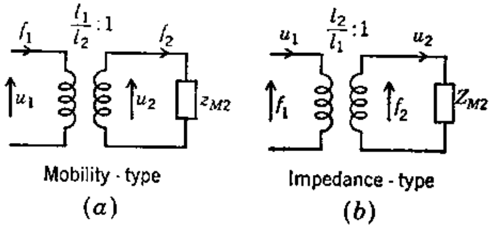


FIG. 3.12. (*a*) Mobility-type and (*b*) impedance-type symbols for a simple lever.

so arranged that a downward force on one end tends to produce an upward force on the other end. An example is shown in Fig. 3.13.

To solve this type of problem, we first write the equations of moments. Summing the moments about the center support gives

$$l_1 f_1 = l_2 f_2$$

and summing the moments about the end support gives

$$(l_1 + l_2) f_1 = l_2 f_3 \tag{3.11}$$

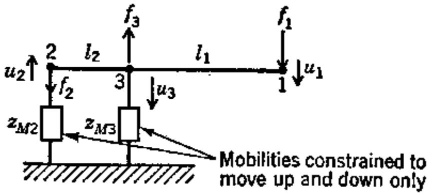


FIG. 3.13. Floating lever.

When the forces are not balanced, and if we assume infinitesimal displacements, the velocities are related to the forces through the mobilities, so that

$$\begin{aligned} u_3 &= z_{M3} f_3 = z_{M3} \frac{l_1 + l_2}{l_2} f_1 \\ u_2 &= z_{M2} f_2 = z_{M2} \frac{l_1}{l_2} f_1 \end{aligned} \tag{3.12}$$

Also, by superposition, it is seen from simple geometry that

$$u_1' = u_3 \frac{l_1 + l_2}{l_2} \qquad \text{for } u_2 = 0$$

$$u_1'' = u_2 \frac{l_1}{l_2} \qquad \text{for } u_3 = 0$$

so that

$$u_1 = u_1' + u_1'' = \frac{l_1 + l_2}{l_2} u_3 + \frac{l_1}{l_2} u_2 \tag{3.13}$$

and, finally,

$$\frac{u_1}{f_1} = z_{M1} = z_{M3}\left(\frac{l_1 + l_2}{l_2}\right)^2 + z_{M2}\left(\frac{l_1}{l_2}\right)^2 \tag{3.14}$$

This equation may be represented by the analogous circuit of Fig. 3.14. The lever loads the generator with two mobilities connected in series, each of which behaves as a simple lever when the other is equal to zero. It will be seen that this is a way of obtaining the equivalent of two series masses without a common zero-velocity (ground) point. This will be illustrated in Example 3.3.

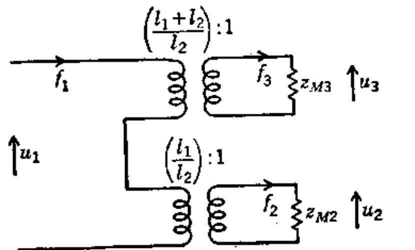


FIG. 3.14. Mobility-type symbol for a floating lever.

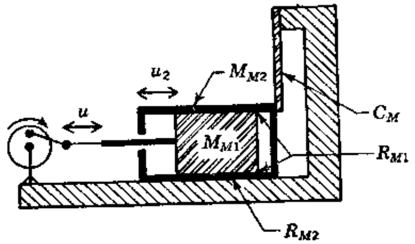


FIG. 3.15. Six-element mechanical device.

Example 3.1. The mechanical device of Fig. 3.15 consists of a piston of mass M_{M1} sliding on an oil surface inside a cylinder of mass M_{M2}. This cylinder in turn slides in an oiled groove cut in a rigid body. The sliding (viscous) resistances are R_{M1} and R_{M2}, respectively. The cylinder is held by a spring of compliance C_M. The mechanical generator maintains a constant sinusoidal velocity of angular frequency ω, whose rms magnitude is u m/sec. Solve for the force f produced by the generator.

Solution. Although the force will be determined ultimately from an analysis of the mobility-type analogous circuit for this mechanical device, it is frequently useful to draw a mechanical-circuit diagram. This interim step to the desired circuit will be especially helpful to the student who is inexperienced in the use of analogies. Its use virtually eliminates errors from the final circuit.

To draw the mechanical circuit, note first the junction points of two or more elements. This locates all element terminals which move with the same velocity. There are in this example two velocities, u and u_2, in addition to "ground," or zero velocity. These two velocities are represented in the mechanical-circuit diagram by the velocities of two imaginary rigid bars, 1 and 2 of Fig. 3.16, which oscillate in a vertical direction. The circuit drawing is made by attaching all element terminals with velocity u to the first bar and all terminals with velocity u_2 to the second bar. All terminals with zero

velocity are drawn to a ground bar. Note that a mass always has one terminal on ground.† Three elements of Fig. 3.15 have one terminal with the velocity u: the generator, the mass M_{M1}, and the viscous resistance R_{M1}. These are attached to bar 1. Four elements have one terminal with the velocity u_2: the viscous resistances R_{M1} and R_{M2}, the mass M_{M2}, and the compliance C_M. These are attached to bar 2. Five elements have one terminal with zero velocity: the generator, both masses, the viscous resistance R_{M2}, and the compliance C_M.

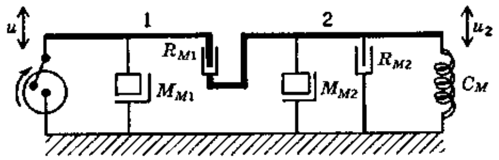


FIG. 3.16. Mechanical circuit for the device of Fig. 3.15.

We are now in a position to transform the mechanical circuit into a mobility-type analogous circuit. This is accomplished simply by replacing the mechanical elements with the analogous mobility-type elements. The circuit becomes that shown in Fig. 3.17. Remember that, in the mobility-type analogy, force "flows" through the elements and velocity is the drop across them. The resistors must have lower case r's written alongside them. As defined above, $r_M = 1/R_M$, and the unit is the mks mechanical mohm.

The equations for this circuit are found in the usual manner, using the rules of Table 3.1. Let us determine $z_M = u/f$, the mechanical mobility presented to the

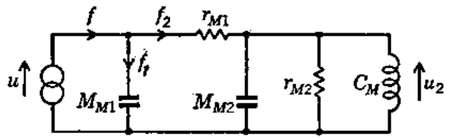


FIG. 3.17. Mobility-type analogous circuit for the device of Fig. 3.15.

generator. The mechanical mobility of the three elements in parallel on the right-hand side of the schematic diagram is

$$\frac{u_2}{f_2} = \frac{1}{\dfrac{1}{1/j\omega M_{M2}} + \dfrac{1}{r_{M2}} + \dfrac{1}{j\omega C_M}}$$

$$= \frac{1}{j\omega M_{M2} + R_{M2} + \dfrac{1}{j\omega C_M}}$$

Including the element r_{M1} the mechanical mobility for that part of the circuit through which f_2 flows is, then,

$$\frac{u}{f_2} = r_{M1} + \frac{1}{j\omega M_{M2} + R_{M2} + \dfrac{1}{j\omega C_M}}$$

Note that the input mechanical mobility z_M is given by

$$z_M = \frac{u}{f} = \frac{u}{f_1 + f_2}$$

† An exception to this rule may occur when the mechanical device embodies one or more floating levers, as we just learned.

and

$$f_1 = \frac{u}{1/j\omega M_{M1}} = j\omega M_{M1} u$$

Substituting f_1 and f_2 into the second equation preceding gives us the input mobility.

$$z_M = \frac{u}{f} = \cfrac{1}{j\omega M_{M1} + \cfrac{1}{r_{M1} + \cfrac{1}{j\omega M_{M2} + R_{M2} + \cfrac{1}{j\omega C_M}}}} \qquad (3.15a)$$

The mechanical impedance is the reciprocal of Eq. (3.15*a*).

$$Z_M = \frac{f}{u} = j\omega M_{M1} + \cfrac{1}{r_{M1} + \cfrac{1}{j\omega M_{M2} + R_{M2} + \cfrac{1}{j\omega C_M}}} \qquad (3.15b)$$

The result is

$$f = Z_M u \qquad \text{newtons} \qquad (3.16)$$

Example 3.2. As a further example of a mechanical circuit, let us consider the two masses of 2 and 4 kg shown in Fig. 3.18. They are assumed to rest on a frictionless

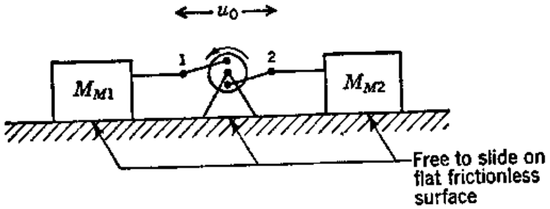


Fig. 3.18. Three-element mechanical device.

plane surface and to be connected together through a generator of constant velocity that is also free to slide on the frictionless plane surface. Let its velocity be

$$u_0 = 2 \cos 1000t \qquad \text{cm/sec}$$

Draw the mobility-type analogous circuit, and determine the force f produced by the generator. Also, determine the mobility presented to the generator.

Solution. The masses do not have the same velocity with respect to ground. The difference between the velocities of the two masses is u_0. The element representing a mass is that shown in Fig. 3.2*a* with one end grounded and the other moving at the velocity of the mass.

The mobility-type circuit for this example is shown in Fig. 3.19. The velocity u_0 equals $u_1 + u_2$, where u_1 is the velocity *with respect to ground* of M_{M1}, and u_2 is that for M_{M2}. The force f is

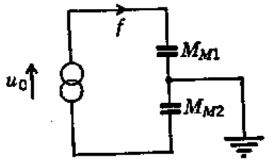


Fig. 3.19. Mobility-type analogous circuit for the device of Fig. 3.18.

$$f_{\text{rms}} = \frac{(u_0)_{\text{rms}}}{(1/j\omega M_{M1}) + (1/j\omega M_{M2})} = \frac{j\omega M_{M1} M_{M2}}{M_{M1} + M_{M2}} u_0$$
$$= \frac{j1000 \times 2 \times 4 \times 0.02}{(2+4)\sqrt{2}} = j18.9 \text{ newtons} \qquad (3.17)$$

The j indicates that the time phase of the force is 90° leading with respect to that of the velocity of the generator.

Obviously, when one mass is large compared with the other, the force is that necessary to move the smaller one alone. This example reveals the only type of case in which masses can be in series without the introduction of floating levers. At most, only two masses can be in series because a common ground is necessary.

The mobility presented to the generator is

$$z_M = \left(\frac{u_0}{f}\right)_{\text{rms}} = \frac{M_{M1} + M_{M2}}{j\omega M_{M1} M_{M2}}$$
$$= \frac{6}{j1000 \times 8} = -j7.5 \times 10^{-4} \text{ mohms} \qquad (3.18)$$

Example 3.3. An example of a mechanical device embodying a floating lever is shown in Fig. 3.20. The masses attached at points 2 and 3 may be assumed to be

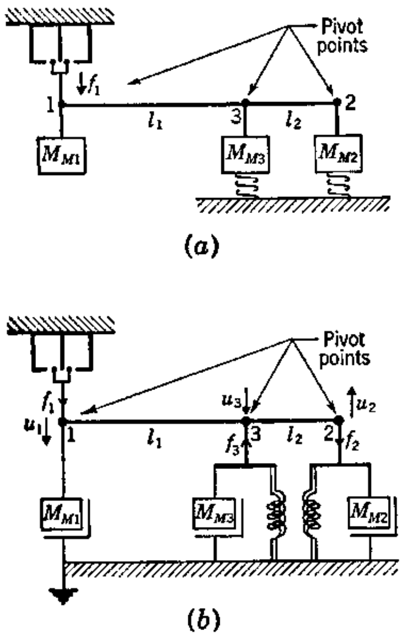


Fig. 3.20. (*a*) Mechanical device embodying a floating lever. (*b*) Mechanical diagram of (*a*). The compliances of the springs are very large so that all of f_2 and f_3 go to move M_{M2} and M_{M3}.

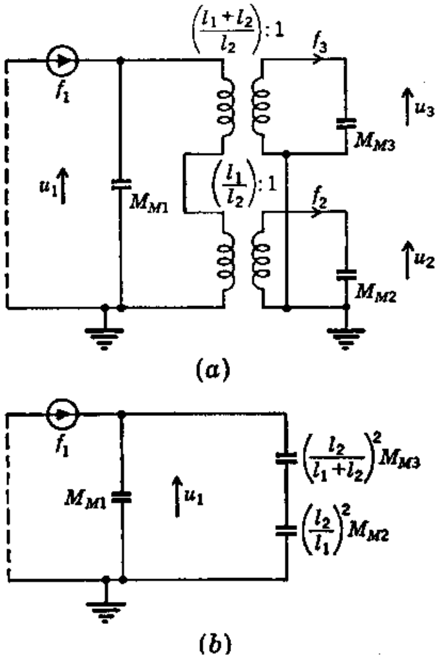


Fig. 3.21. (*a*) Mobility-type analogous circuit for the device of Fig. 3.20. (*b*) Same as (*a*) but with transformers removed.

resting on very compliant springs. The driving force f_1 will be assumed to have a frequency well above the resonance frequencies of the masses and their spring supports so that

$$z_{M2} \doteq \frac{1}{j\omega M_{M2}}$$
$$z_{M3} \doteq \frac{1}{j\omega M_{M3}}$$

Also, assume that a mass is attached to the weightless lever bar at point 1, with a mobility

$$z_{M1} = \frac{1}{j\omega M_{M1}}$$

Solve for the total mobility presented to the constant-force generator f_1.

Solution. By inspection, the mobility-type analogous circuit is drawn as shown in Fig. 3.21*a* and *b*. Solving for $z_M = u_1/f_1$, we get

$$z_M = \frac{1}{j\omega\left[\dfrac{M_{M2}M_{M3}l_2^2}{M_{M3}l_1^2 + M_{M2}(l_1 + l_2)^2} + M_{M1}\right]} \tag{3.19}$$

Note that if $l_2 \to 0$, the mobility is simply that of the mass M_{M1}. Also, if $l_1 \to 0$, the mobility is that of M_{M1} and M_{M3}, that is,

$$z_M = \frac{1}{j\omega(M_{M3} + M_{M1})} \tag{3.20}$$

It is possible with one or more floating levers to have one or more masses with no ground terminal.

PART VII *Acoustical Circuits*

3.4. Acoustical Elements. Acoustical circuits are frequently more difficult to draw than mechanical ones because the elements are less easy to identify. As was the case for mechanical circuits, the more obvious forms of the elements will be useful as an intermediate step toward drawing the analogous circuit diagram. When the student is more familiar with acoustical circuits, he will be able to pass directly from the acoustic device to the final form of the equivalent circuit.

In acoustic devices, the quantity we are able to measure most easily without modification of the device is sound pressure. Such a measurement is made by inserting a small hollow probe tube into the sound field at the desired point. This probe tube leads to one side of a microphone diaphragm. The other side of the diaphragm is exposed to atmospheric pressure. A movement of the diaphragm takes place when there is a difference in pressure across it. This difference between atmospheric pressure and the pressure with the sound field is the sound pressure p.

Because we can measure sound pressure by such a probe-tube arrangement without disturbing the device, it seems that sound pressure is analogous to voltage in electrical circuits. Such a choice requires us to consider current as being analogous to some quantity which is proportional to velocity. As we shall show shortly, a good choice is to make current analogous to volume velocity, the volume of gas displaced per second.

A strong argument can be made for this choice of analogy when one considers the relations governing the flow of air inside such acoustic devices as loudspeakers, microphones, and noise filters. Inside a certain type of microphone, for example, there is an air cavity that connects to the outside air through a small tube (see Fig. 3.22). Assume, now, that the outer end of this tube is placed in a sound wave. The wave will cause a movement of the air particles in the tube. Obviously, there is a junction between the tube and the cavity at the inner end of the tube at point A. Let us ask ourselves the question, What physical quantities are continuous at this junction point?

First, the sound pressure just inside the tube at A is the same as that in the cavity just outside A. That is to say, we have continuity of sound pressure. Second, the quantity of air leaving the inner end of the small tube in a given interval of time is the quantity that enters the cavity in the same interval of time. That is, the mass per second of gas leaving the small tube equals the mass per second of gas entering the volume. Because the pressure is the same at both places, the density of the gas must also be the same, and it follows that there is continuity of volume velocity (cubic meters per second) at this junction. Analogously, in the case of electricity, there is continuity of electric current at a junction. Continuity of volume velocity must exist even if there are several tubes or cavities joining near one point. A violation of the law of conservation of mass otherwise would occur.

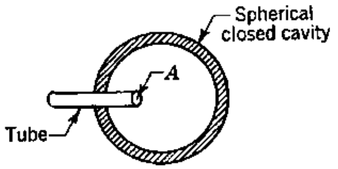


FIG. 3.22. Closed cavity connecting to the outside air through a tube of cross-sectional area S. The junction plane between the tube and the cavity occurs at A.

We conclude that the quantity that flows *through* our acoustical elements must be the volume velocity U in cubic meters per second and the drop across our acoustical elements must be the pressure p in newtons per square meter. This conclusion indicates that the impedance type of analogy is the preferred analogy for acoustical circuits. The product of the effective sound pressure p times the in-phase component of the effective volume velocity U gives the acoustic power in watts.

In this part, we shall discuss the more general aspects of acoustical circuits. In Chap. 5 of this book, we explain fully the approximations involved and the rules for using the concepts enunciated here in practical problems.

Acoustic Mass M_A. Acoustic mass is a quantity proportional to mass but having the dimensions of kilograms per meter[4]. It is associated with a mass of air accelerated by a net force which acts to displace the gas without appreciably compressing it. The concept of acceleration without compression is an important one to remember. It will assist you in distinguishing acoustic masses from other elements.

The acoustical element that is used to represent an acoustic mass is a tube filled with the gas as shown in Fig. 3.23.

The physical law governing the motion of a mass that is acted on by a force is Newton's second law, $f(t) = M_M\, du(t)/dt$. This law may be expressed in acoustical terms as follows,

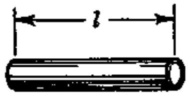


FIG. 3.23. Tube of length l and cross-sectional area S.

$$\frac{f(t)}{S} = \frac{M_M}{S}\frac{d[u(t)S]}{dt\,S} = p(t) = \frac{M_M}{S^2}\frac{dU(t)}{dt}$$

or

$$p(t) = M_A \frac{dU(t)}{dt} \tag{3.21}$$

where $p(t)$ = instantaneous difference between pressures in newtons per square meter existing at each end of a mass of gas of M_M kg undergoing acceleration.

$M_A = M_M/S^2$ = acoustic mass in kilograms per meter[4] of the gas undergoing acceleration. This quantity is nearly equal to the mass of the gas inside the containing tube divided by the square of the cross-sectional area. To be more exact we must note that the gas in the immediate vicinity of the ends of the tube also adds to the mass. Hence, there are "end corrections" which must be considered. These corrections are discussed in Chap. 5 (pages 132 to 139).

$U(t)$ = instantaneous volume velocity, of the gas in cubic meters per second across any cross-sectional plane in the tube. The volume velocity $U(t)$ is equal to the linear velocity $u(t)$ multiplied by the cross-sectional area S.

In the steady state, with an angular frequency ω, we have

$$p = j\omega M_A U \tag{3.22}$$

where p and U are taken to be rms complex quantities.

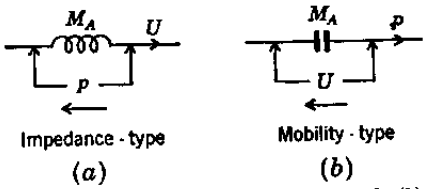


FIG. 3.24. (a) Impedance-type and (b) mobility-type symbols for an acoustic mass.

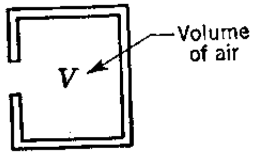


FIG. 3.25. Enclosed volume of air V with opening for entrance of pressure variations.

The impedance-type analogous symbol for acoustic mass is shown in Fig. 3.24a, and the mobility-type is given in Fig. 3.24b. In the steady state, for either, we get Eq. (3.22). The arrows point in the direction of positive flow or positive drop.

Acoustic Compliance C_A. Acoustic compliance is a constant quantity having the dimensions of meter[5] per newton. It is associated with a volume of air that is compressed by a net force without an appreciable average displacement of the center of gravity of air in the volume. In other words, compression without acceleration identifies an acoustic compliance.

The acoustical element that is used to represent an acoustic compliance is a volume of air drawn as shown in Fig. 3.25.

The physical law governing the compression of a volume of air being acted on by a net force was given as $f(t) = (1/C_M)\int u(t)\, dt$. Converting from mechanical to acoustical terms,

$$\frac{f(t)}{S} = \frac{1}{C_M S}\int u(t)\frac{S}{S}\,dt \qquad \text{or} \qquad p(t) = \frac{1}{C_M S^2}\int U(t)\,dt$$

or

$$p(t) = \frac{1}{C_A}\int U(t)\,dt \tag{3.23}$$

where $p(t)$ = instantaneous pressure in newtons per square meter acting to compress the volume V of the air.

$C_A = C_M S^2$ = acoustic compliance in meters[5] per newton of the volume of the air undergoing compression. The acoustic compliance is nearly equal to the volume of air divided by γP_0, as we shall see in Chap. 5 (pages 128 to 131).

$U(t)$ = instantaneous volume velocity in cubic meters per second of the air flowing into the volume that is undergoing compression. The volume velocity $U(t)$ is equal to the linear velocity $u(t)$ multiplied by the cross-sectional area S.

In the steady state with an angular frequency ω, we have

$$p = \frac{U}{j\omega C_A} \tag{3.24}$$

where p and U are taken to be rms complex quantities.

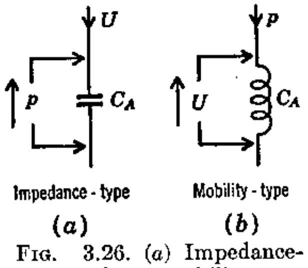


FIG. 3.26. (a) Impedance-type and (b) mobility-type symbols for an acoustic compliance.

The impedance-type analogous element for acoustic compliance is shown in Fig. 3.26a and the mobility-type in Fig. 3.26b. In the steady state for either, Eq. (3.24) applies.

Acoustic Resistance R_A, and Acoustic Responsiveness r_A. Acoustic resistance R_A is associated with the dissipative losses occurring when there is a viscous movement of a quantity of gas through a fine-mesh screen or through a capillary tube. It is a constant quantity having the dimensions newton-seconds per meter[5]. The unit is the mks acoustic ohm.

FIG. 3.27. Fine-mesh screen which serves as an acoustical symbol for acoustic resistance.

The acoustic element used to represent an acoustic resistance is a fine-mesh screen drawn as shown in Fig. 3.27.

The reciprocal of acoustic resistance is the acoustic responsiveness r_A. The unit is the mks acoustic mohm with dimensions meter^5 per second per newton.

The physical law governing dissipative effects in a mechanical system was given by $f(t) = R_M u(t)$, or, in terms of acoustical quantities,

$$p(t) = R_A U(t) = \frac{1}{r_A} U(t) \tag{3.25}$$

where $p(t)$ = difference between instantaneous pressures in newtons per square meter across the dissipative element. In the steady state p is an rms complex quantity.

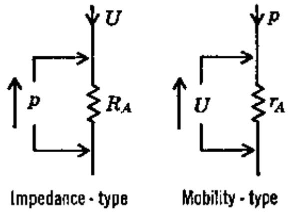
$R_A = R_M/S^2$ = acoustic resistance in acoustic ohms, *i.e.*, newton-seconds per meter^5.

$r_A = r_M S^2$ = acoustic responsiveness in acoustic mohms, *i.e.*, meter^5 per newton-seconds.

$U(t)$ = instantaneous volume velocity in cubic meters per second of the gas through the cross-sectional area of resistance. In the steady state U is an rms quantity.

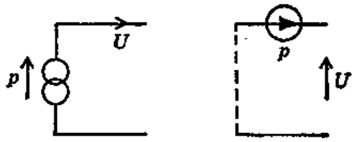
The impedance-type analogous symbol for acoustic resistance is shown in Fig. 3.28*a* and the mobility-type in Fig. 3.28*b*.

Acoustic Generators. Acoustic generators can be of either the constant-volume velocity or the constant-pressure type. The prime movers in our



(*a*) (*b*)

FIG. 3.28. (*a*) Impedance-type symbol for acoustic resistance and (*b*) mobility-type symbol for acoustic responsiveness.



(*a*) (*b*)

FIG. 3.29. (*a*) Impedance-type and (*b*) mobility-type symbols for a constant-pressure generator.

acoustical circuits will be exactly like those shown in Figs. 3.7 and 3.9 except that u_2 often will be zero and u_1 will be the velocity of a small piston of area S. Remembering that $u = u_1 - u_2$, we see that the generator of Fig. 3.7 has a constant-volume velocity $U = uS$ and that of Fig. 3.9 a constant pressure of $p = f/S$.

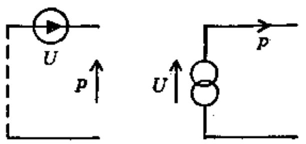
The two types of analogous symbols for acoustic generators are given in Figs. 3.29 and 3.30. The arrows point in the direction of the positive

terminal or the positive flow. As before, the double circles indicate zero impedance or mobility and a dashed line infinite impedance or mobility.

Mechanical Rotational Systems. Mechanical rotational systems are handled in the same manner as mechanical rectilineal systems. The following quantities are analogous in the two systems.

Rectilineal systems	*Rotational systems*
f = force, newtons	T = torque, newton-m
v = velocity, m/sec	θ = angular velocity, radians/sec
ξ = displacement, m	ϕ = angular displacement, radians
$Z_M = f/v$ = mechanical impedance, mks mechanical ohms	$Z_R = T/\theta$ = rotational impedance, mks rotational ohms
$z_M = v/f$ = mechanical mobility, mks mechanical mohms	$z_R = \theta/T$ = rotational mobility, mks rotational mohms
R_M = mechanical resistance, mks mechanical ohms	R_R = rotational resistance, mks rotational ohms
r_M = mechanical responsiveness, mks mechanical mohms	r_R = rotational responsiveness, mks rotational mohms
M_M = mass, kg	I_R = moment of inertia, kg-m^2
C_M = mechanical compliance, m/newton	C_R = rotational compliance, radians/newton-m
W_M = mechanical power, watts	W_R = rotational power, watts

Example 3.4. The acoustic device of Fig. 3.31 consists of three cavities V_1, V_2, and V_3, two fine-mesh screens R_{A1} and R_{A2}, four short lengths of tube T_1, T_2, T_3, and T_4, and a constant-pressure generator. Because the air in the tubes is not confined, it



(*a*) (*b*)

FIG. 3.30. (*a*) Impedance-type and (*b*) mobility-type symbols for a constant-volume velocity generator.

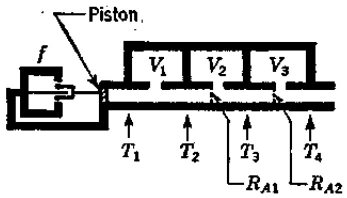


FIG. 3.31. Acoustic device consisting of four tubes, three cavities, and two screens driven by a constant-pressure generator.

experiences negligible compression. Because the air in each of the cavities is confined, it experiences little average movement. Let the force of the generator be

$$f = 10^{-5} \cos 1000t \qquad \text{newtons}$$

the radius of the tube $a = 0.5$ cm; the length of each of the four tubes $l = 5$ cm; the volume of each of the three cavities $V = 10$ cm^3; and the magnitude of the two acoustic resistances $R_A = 10$ mks acoustic ohms. Neglecting end corrections, solve for the volume velocity U_0 at the end of the tube T_4.

Solution. Remembering that there is continuity of volume velocity and pressure at the junctions, we can draw the impedance-type analogous circuit from inspection. It is shown in Fig. 3.32. The bottom line of the schematic diagram represents atmospheric pressure, which means that here the variational pressure p is equal to zero. At each of the junctions of the elements 1 to 4, a different variational pressure can be observed. The end of the fourth tube (T_4) opens to the atmosphere, which requires that M_{A4} be connected directly to the bottom line of Fig. 3.32.

Note that the volume velocity of the gas leaving the tube T_1 is equal to the sum of the volume velocities of the gas entering V_1 and T_2. The volume velocity of the gas leaving T_2 is the same as that flowing through the screen R_{A1} and is equal to the sum of the volume velocities of the gas entering V_2 and T_3.

One test of the validity of an analogous circuit is its behavior for direct current. If one removes the piston and blows into the end of the tube T_1 (Fig. 3.31), a steady flow of air from T_4 is observed. Some resistance to this flow will be offered by the two screens R_{A1} and R_{A2}. Similarly in the schematic diagram of Fig. 3.32, a steady pressure p will produce a steady flow U through M_{A4}, resisted only by R_{A1} and R_{A2}.

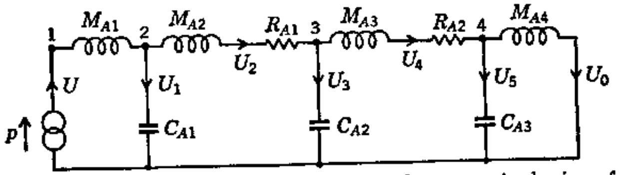


FIG. 3.32. Impedance-type analogous circuit for the acoustic device of Fig. 3.31.

As an aside, let us note that an acoustic compliance can occur in a circuit without one of the terminals being at ground potential only if it is produced by an elastic diaphragm. For example, if the resistance R_{A1} in Fig. 3.31 were replaced by an impervious but elastic diaphragm, the element R_{A1} in Fig. 3.32 would be replaced by a compliance-type element with both terminals above ground potential. In this case a steady flow of air could not be maintained through the device of Fig. 3.31, as can also be seen from the circuit of Fig. 3.32, with R_{A1} replaced by a compliance.

Determine the element sizes of Fig. 3.32.

$$p = \frac{f}{S} = \frac{10^{-6}\cos 1000t}{\pi(5 \times 10^{-3})^2} = 0.1273 \cos 1000t \qquad \text{newton/m}^2$$

$$M_{A1} = M_{A2} = M_{A3} = M_{A4} = \frac{\rho_0 l}{S} = \frac{1.18 \times 0.05}{7.85 \times 10^{-5}} = 750 \text{ kg/m}^4$$

$$C_{A1} = C_{A2} = C_{A3} = \frac{V}{\gamma P_0} = \frac{10^{-6}}{1.4 \times 10^5} = 7.15 \times 10^{-11} \text{ m}^5\text{/newton}$$

$$R_{A1} = R_{A2} = 10 \text{ mks acoustic ohms}$$

As is customary in electric-circuit theory, we solve for U_0 indirectly. First, arbitrarily let $U_0 = 1$ m³/sec, and determine the ratio p/U_0.

$$\begin{aligned}
p_4 &= j\omega M_{A4}U_0 = j7.5 \times 10^5 \text{ newtons/m}^2\\
U_5 &= j\omega C_{A3}p_4 = -5.36 \times 10^{-2} \text{ m}^3\text{/sec}\\
U_4 &= U_5 + U_0 = 0.946\\
p_3 &= (R_{A2} + j\omega M_{A3})U_4 + p_4 = 9.46 + j14.6 \times 10^5\\
U_3 &= j\omega C_{A2}p_3 = -0.1043 + j6.77 \times 10^{-7}\\
U_2 &= U_3 + U_4 = 0.842 + j6.77 \times 10^{-7}\\
p_2 &= (R_{A1} + j\omega M_{A2})U_2 + p_3 = 17.37 + j2.091 \times 10^6\\
U_1 &= j\omega C_{A1}p_2 = -0.1496 + j1.242 \times 10^{-6}\\
U &= U_2 + U_1 = 0.692 + j1.919 \times 10^{-6}\\
p &= j\omega M_{A1}U + p_2 = 15.93 + j2.61 \times 10^6 = \frac{p}{U_0} \qquad \text{for } U_0 = 1
\end{aligned}$$

The desired value of U_0 is

$$\begin{aligned}
U_0 = p\frac{U_0}{p} &= \frac{0.1273 \cos 1000t}{15.93 + j2.61 \times 10^6}\\
&\doteq 4.88 \times 10^{-8} \cos (1000t - 90°)\\
&\doteq 4.88 \times 10^{-8} \sin 1000t
\end{aligned}$$

In other words, the impedance is principally that of the four acoustic masses in series so that U_0 lags p by nearly 90°.

Example 3.5. A Helmholtz resonator is frequently used as a means for eliminating an undesired frequency component from an acoustic system. An example is given in Fig. 3.33*a*. A constant-force generator G produces a series of tones, among which is one that is not wanted. These tones actuate a microphone M whose acoustic impedance is 500 mks acoustic ohms. If the tube T has a cross-sectional area of 5 cm², $l_1 = l_2 = 5$ cm, $l_3 = 1$ cm, $V = 1000$ cm³, and the cross-sectional area of l_3 is 2 cm², what frequency is eliminated from the system?

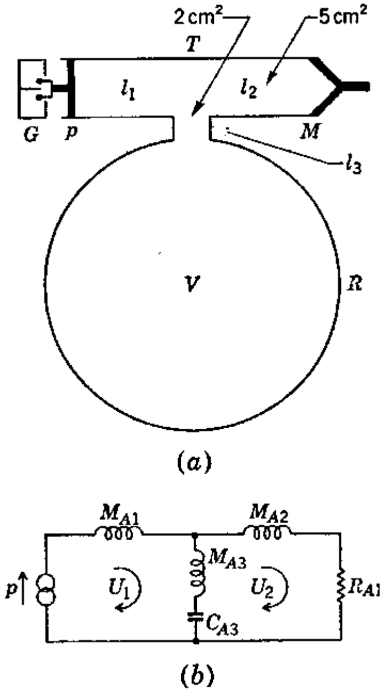


FIG. 3.33. (*a*) Acoustic device consisting of a constant-force generator G, piston P, tube T with length $l_1 + l_2$, microphone M, and Helmholtz resonator R with volume V and connecting tube as shown. (*b*) Impedance-type analogous circuit for the device of (*a*).

Solution. By inspection we may draw the impedance-type analogous circuit of Fig. 3.33*b*. The element sizes are

$$M_{A1} = M_{A2} = \frac{\rho_0 l_1}{S_T} = \frac{1.18 \times 0.05}{5 \times 10^{-4}} = 118 \text{ kg/m}^4$$

$$M_{A3} = \frac{\rho_0 l_3}{S_R} = \frac{1.18 \times 0.01}{2 \times 10^{-4}} = 59 \text{ kg/m}^4$$

$$C_{A3} = \frac{V}{\gamma P_0} = \frac{10^{-3}}{1.4 \times 10^5} = 7.15 \times 10^{-9} \text{ m}^5\text{/newton}$$

$$R_{A1} = 500 \text{ mks acoustic ohms}$$

It is obvious that the volume velocity U_2 of the transducer M will be zero when the shunt branch is at resonance. Hence,

$$\omega = \frac{1}{\sqrt{M_{A3}C_{A3}}} = \frac{10^4}{\sqrt{42.2}} = 1540 \text{ radians/sec}$$

$$f = 245 \text{ cps}$$