ple, we may be able to determine the rest of the elements by requiring that the transfer function of the system be correct.

**Exercises 2.4** *2.4-1.* Show that the system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a & -b & -c \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [1 \quad 0 \quad 0]\mathbf{x}(t)$$

is controllable and observable for all values of $a$, $b$, and $c$. Find the transfer-function representation of this system.

*answer:*

$$G(s) = \frac{1}{s^3 + cs^2 + bs + a}$$

*2.4-2.* Investigate the controllability of the following systems. If the system is controllable and observable, find its transfer-function representation. If the system is either uncontrollable or unobservable, find the number of states which are controllable and the number which are observable.

(*a*) $\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$ $\quad y = [1 \quad 1]\mathbf{x}$

(*b*) $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$ $\quad y = [0 \quad 1 \quad 1]\mathbf{x}$

(*c*) $\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$ $\quad y = [1 \quad 0 \quad 1]\mathbf{x}$

*answers:*

(*a*) Controllable and observable,

$$\frac{y(s)}{u(s)} = \frac{s + 1}{s^2 - s - 1}$$

(*b*) Controllable but unobservable; only one state is observable
(*c*) Observable but uncontrollable; only two states are controllable



## 2.5 *Phase variables*

The use of phase variables to describe a control system is simple, and hence much work has been done in this particular coordinate system. The phase variables are defined as those particular state variables which are obtained from one of the system variables and its $n - 1$ derivatives. Often the variable used is the system output, and the remaining state variables are then the $n - 1$ derivatives of the output. In a third-order positioning system, for example, the output might be $\theta$, so that $x_1 = \theta$, $x_2 = \dot{\theta} = \omega$, and $x_3 = \dot{\omega}$.

As a specific example, consider the control system represented by the block diagram of Fig. 2.5-1. The transfer function for this example is

$$G(s) = \frac{y(s)}{u(s)} = \frac{K}{s^3 + a_3 s^2 + a_2 s + a_1}$$

By cross multiplying this becomes

$$(s^3 + a_3 s^2 + a_2 s + a_1)y(s) = Ku(s) \tag{2.5-1}$$

Since, by definition, a transfer function assumes zero initial conditions, Eq. (2.5-1) is simply the Laplace transform of a third-order differential equation with zero initial conditions. It is a simple matter to reconstruct the original differential equation by identifying derivatives with powers of $s$, so that in the time domain Eq. (2.5-1) becomes

$$\dddot{y}(t) + a_3\ddot{y}(t) + a_2\dot{y}(t) + a_1 y(t) = Ku(t) \tag{2.5-2}$$

In order to express Eq. (2.5-2) in phase variables as three first-order differential equations, let $x_1 = y(t)$, and then according to the definition of the phase variables, $x_2 = \dot{y}(t)$, and $x_3 = \ddot{y}(t)$. If these substitutions are made in Eq. (2.5-2), the result is

$$\dot{x}_3 + a_3 x_3 + a_2 x_2 + a_1 x_1 = Ku \tag{2.5-3}$$

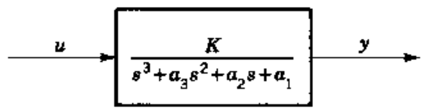


***Fig. 2.5-1*** System to be represented in phase variables.

and the three first-order differential equations are

$$\begin{aligned} \dot{x}_1 &= x_2 \qquad \dot{x}_2 = x_3 \\ \dot{x}_3 &= -a_1x_1 - a_2x_2 - a_3x_3 + Ku \end{aligned} \tag{2.5-4}$$

where

$$y = x_1 \tag{2.5-5}$$

These equations are of the form (**Ab**) and (**c**), where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix} \qquad \text{and} \qquad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and the desired system representation has been achieved.

In the $n$th-order case, where $G(s)$ is

$$\frac{y(s)}{u(s)} = G(s) = \frac{K}{s^n + a_ns^{n-1} + a_{n-1}s^{n-2} + \cdots + a_2s + a_1}$$

the resulting $n$th-order differential equation corresponding to Eq. (2.5-2) is

$$y^{(n)}(t) + a_ny^{(n-1)}(t) + \cdots + a_2\dot{y}(t) + a_1y(t) = Ku(t) \tag{2.5-6}$$

If we now let $x_1 = y$ and $x_2 = \dot{x}_1, \ldots, x_n = \dot{x}_{n-1}$, then Eq. (2.5-6) becomes

$$\dot{x}_n + a_nx_n + a_{n-1}x_{n-1} + \cdots + a_2x_2 + a_1x_1 = Ku$$

The phase-variable representation for this system is then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\cdots\cdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_1x_1 - a_2x_2 - \cdots - a_{n-1}x_{n-1} - a_nx_n + Ku \end{aligned} \tag{2.5-7}$$

with

$$y = x_1 \tag{2.5-8}$$



The matrices **A**, **b**, and **c** are then

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & -a_n \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ K \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

The block diagrams for the phase-variable representations of both the third-order example and the $n$th-order system are shown in Fig. 2.5-2. A simple examination of these block diagrams, the original transfer function or $n$th-order differential equations, and the phase-variable representation reveals the fact that any of these means of representation may be

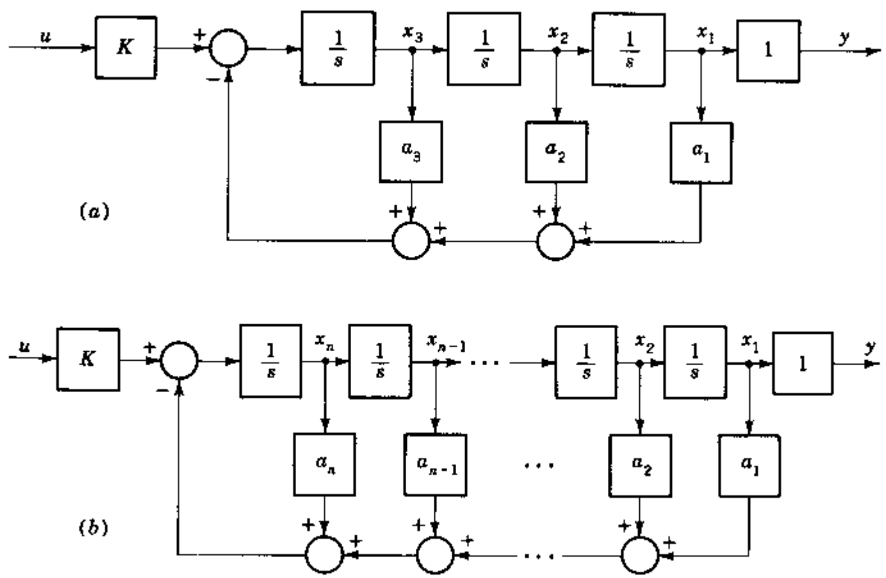


*Fig. 2.5-2* Block diagrams for the phase-variable representation. (*a*) A third-order example; (*b*) an $n$th-order system.

determined from the other *by inspection*. This is very convenient in situations like that in Chap. 8, where it is desired to transfer one method of representation to the other easily. The reader familiar with analog-computer techniques will note the similarity of the phase-variable representation as illustrated by Fig. 2.5-2 with the direct programming method of simulating transfer functions.

An alternate block diagram for the phase-variable representation is shown in Fig. 2.5-3. This block-diagram form is suggested by a knowledge of the transfer function $G(s)$. Although the phase-variable equations are not so obvious in this alternate block diagram, we shall nevertheless find use for both forms of block diagram in the later chapters.

For the reader who has had little acquaintance with the block-diagram approach to control-system representation, it is pointed out that this development could just as easily have been based on a discussion of the differential equation (2.5-2) or (2.5-6). Since any set of linear simultaneous ordinary differential equations can be arranged as one $n$th-order equation, the approach from the point of view of either the block diagram or Eq. (2.5-2) or (2.5-6) is completely general and equivalent.

So far we have considered only the case where the transfer function $G(s)$ has no zeros. In order to include zeros in $G(s)$, it is necessary to modify the above approach slightly. To see why this is necessary, let us consider the same third-order example except with an added zero, so that

$$\frac{y(s)}{u(s)} = G(s) = \frac{K(c_2 s + c_1)}{s^3 + a_3 s^2 + a_2 s + a_1} \tag{2.5-9}$$

If we proceed as before by letting $x_1 = y$, $x_2 = \dot{y}$, and $x_3 = \ddot{y}$, then Eq. (2.5-3) becomes for this case

$$\dot{x}_3 + a_3 x_3 + a_2 x_2 + a_1 x_1 = K(c_2 \dot{u} + c_1 u)$$

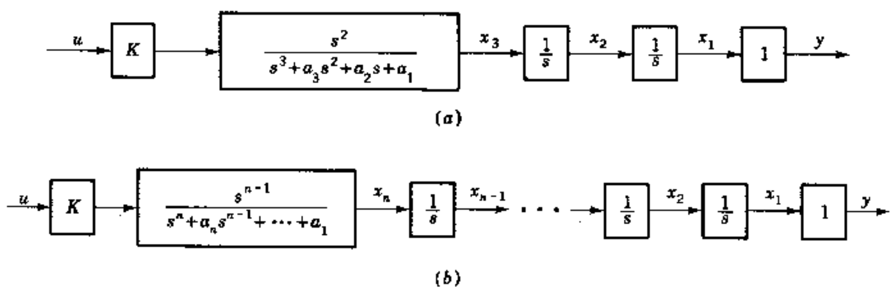


*Fig. 2.5-3* Alternate block diagrams for the systems shown in Fig. 2.5-2. (*a*) A third-order example; (*b*) an $n$th-order system.



The phase-variable representation of the plant equation as given by Eqs. (2.5-4) now contains a $\dot{u}$ term on the right-hand side, which violates the assumed form of Eq. (**Ab**).

In order to avoid this problem, let the transfer function $G(s)$ be divided into two parts in the following manner, as shown in Fig. 2.5-4,

$$G(s) = \frac{y(s)}{u(s)} = \frac{x_1(s)}{u(s)}\frac{y(s)}{x_1(s)}$$

where

$$\frac{x_1(s)}{u(s)} = \frac{K}{s^3 + a_3 s^2 + a_2 s + a_1}$$

and

$$\frac{y(s)}{x_1(s)} = c_2 s + c_1$$

The first transfer function, $x_1(s)/u(s)$, is identical to the original transfer function without the zero, and therefore its phase-variable representation is given by Eqs. (2.5-4). The second transfer function, $y(s)/x_1(s)$, however, indicates that $y$ is no longer equal to just $x_1$ but is now

$$y(t) = c_2\dot{x}_1(t) + c_1 x_1(t) = c_2 x_2(t) + c_1 x_1(t)$$

where the second expression is written by using the fact that $\dot{x}_1 = x_2$ since the system is in phase variables.

The complete phase-variable description of the system represented by the transfer function (2.5-9) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix} u$$

$$y = [c_1 \quad c_2 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

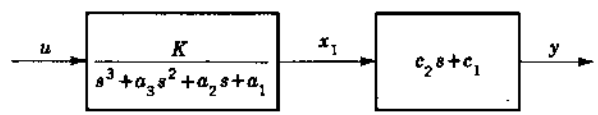


*Fig. 2.5-4* Technique for handling a system with a zero.

A comparison of this result with the representation of the original system without the zero indicates that the only change made by the addition of the zero is in the output expression.

In a similar fashion we can show that the phase-variable representation of the general case where $G(s)$ has $m - 1$ zeros and $n$ poles, $m \leq n$,

$$G(s) = \frac{K(c_m s^{m-1} + c_{m-1}s^{m-2} + \cdots + c_2 s + c_1)}{s^n + a_n s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_2 s + a_1} \tag{2.5-10}$$

is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & -a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ K \end{bmatrix} u \tag{2.5-11}$$

$$y = [c_1 \quad c_2 \quad \cdots \quad c_m \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_{n-1} \\ x_n \end{bmatrix} \tag{2.5-12}$$

Once again we see that the zeros affect only the output expression. The block diagrams of the phase-variable representation of the third-order example and the general $n$th-order system (Fig. 2.5-5) also illustrate this feature. We note, however, that the phase-variable representation is still easily determined by inspection from the transfer function and vice versa.

It should be noted that the specification of a separate gain term $K$ in the transfer function (2.5-10) is somewhat artificial and arbitrary. For example, either $K$ or $c_m$ could be required to be unity without loss of generality. The general form is retained, however, for added flexibility.

Zeros may also be included by modifying the control vector **b** rather than the output vector **c**. In order to illustrate this technique, consider a simple second-order case

$$G(s) = \frac{K(c_2 s + c_1)}{s^2 + a_2 s + a_1} \tag{2.5-13}$$



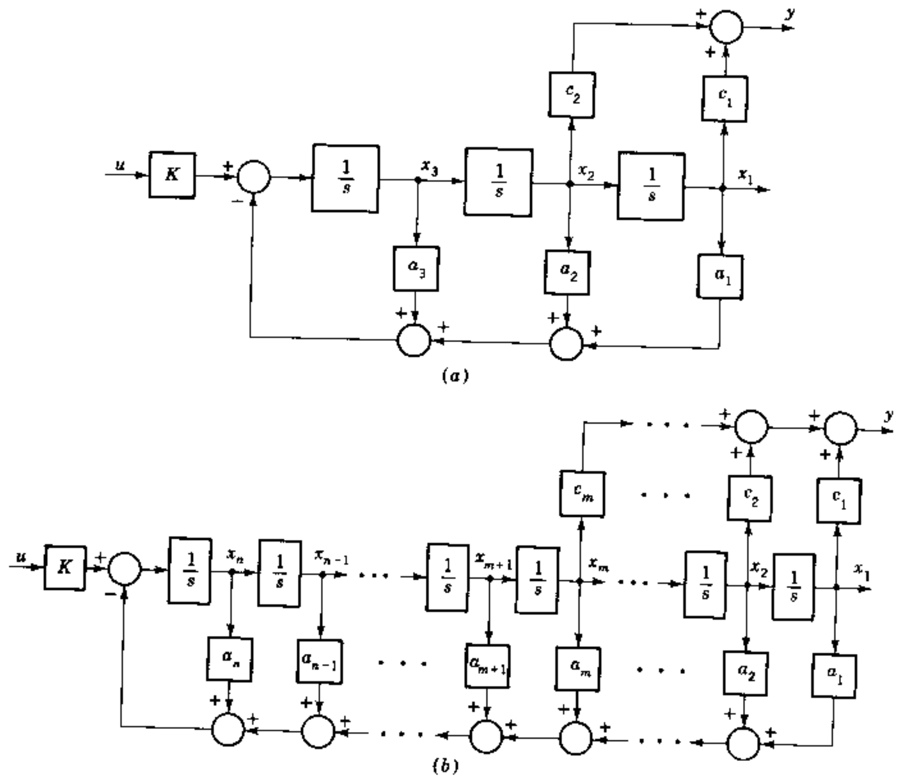


*Fig. 2.5-5* Block diagrams for phase-variable representation. (*a*) A third-order example; (*b*) an $n$th-order system.

We wish to find a phase-variable representation of this system in the following form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u \tag{2.5-14}$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.5-15}$$

Here we have retained the **A** and **c** matrices appropriate for the case without a zero but have chosen a general control vector. The problem is to determine the values of $b_1$ and $b_2$ such that this system represents the original transfer function. For example, if the zero were not present, we know that $b_1 = 0$ and $b_2 = K$ would be the correct answer.

In order to determine the correct values of $b_1$ and $b_2$, the uniqueness property of the transfer-function representation is used. More specifically, $b_1$ and $b_2$ are picked such that the transfer function associated with the phase-variable representation of Eqs. (2.5-14) and (2.5-15) is equal to the original transfer function of Eq. (2.5-13), i.e., we set

$$\mathbf{c}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = G(s) \tag{2.5-16}$$

For the problem at hand this becomes

$$[1 \quad 0]\begin{bmatrix} s & -1 \\ a_1 & s + a_2 \end{bmatrix}^{-1}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{b_1 s + (b_1 a_2 + b_2)}{s^2 + a_2 s + a_1} = \frac{K(c_2 s + c_1)}{s^2 + a_2 s + a_1}$$

Equating like powers of $s$ in the numerator of the two expressions, we obtain

$$b_1 = Kc_2 \qquad b_1 a_2 + b_2 = Kc_1$$

or

$$b_1 = Kc_2 \qquad b_2 = K(c_1 - a_2 c_2)$$

which is the desired result.

Unfortunately, the elements of $\mathbf{b}$ are not simply related to the elements of the transfer function, and therefore the phase-variable representation cannot be determined by inspection from the transfer function. Because of this fact, this alternate approach to the problem of including zeros is seldom used, and use of the first approach is implied whenever we refer to phase variables.

Although phase variables provide a simple means for representing a system in state-variable form as Eqs. (**Ab**) and (**c**), they do, however, have two disadvantages. First, the solution of the resulting $n$ first-order differential equations is no simpler than the solution of the original $n$th-order equation. Hence, our state-variable representation is of no assistance in finding the time response of the system.

Second, phase variables are, in general, not real physical variables and therefore are not available for measurement or manipulation. If $G(s)$ has no zeros, the phase variables are equal to the output and its first $n - 1$ derivatives. Unfortunately, it is very difficult physically to take $n - 1$ derivatives if $n$ is greater than 2 or 3; in the presence of noise it becomes impossible. (It is exactly for this reason that analog computers use integrators rather than differentiators.) If $G(s)$ has zeros, the phase variables bear little resemblance to real physical quantities in the system.

Thus, while the phase variables are simple to realize mathematically, they are not a practical set of state variables from a measurement or control point of view. It is shown in later chapters that for a large class of systems it is actually necessary to feed back not just the output but *all the state variables.* It is quite obvious that, from an engineering point of view, all the state variables must be real and measurable. This does not mean that phase variables are not useful and even valuable means of state-variable representation. We shall make extensive use of phase variables throughout Chap. 8, for example.



Before ending our discussion of phase variables, let us consider briefly the concepts of controllability and observability and their relation to the phase-variable representation of a system. It is not difficult to demonstrate that the phase-variable representation is always controllable and observable (see, for example, Exercise 2.4-1). Hence, phase variables may only be used to represent systems which are controllable and observable. In addition, it is possible to show (Kalman, 1963) that any controllable and observable system may always be represented in phase-variable form.

**Exercises 2.5** *2.5-1.* Describe the following systems in phase variables.

(*a*) $G(s) = \dfrac{K}{s^2 + 2s + 1}$ (*b*) $G(s) = \dfrac{K(s+1)}{s^2 + s + 1}$

(*c*) $G(s) = \dfrac{K(s+2)}{s^3 + 2s^2 + s + 1}$

*answers:*

(*a*) $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ K \end{bmatrix}u \qquad y = x_1$

(*b*) $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ K \end{bmatrix}u \qquad y = x_1 + x_2$

(*c*) $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix}u \qquad y = 2x_1 + x_2$

*2.5-2.* Find the transfer functions for the following systems.

(*a*) $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}u \qquad y = [1 \quad 0 \quad 0]\mathbf{x}$

(*b*) $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & -3 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix}u \qquad y = [4 \quad 1 \quad 0]\mathbf{x}$

*answers:*

(a) $G(s) = \dfrac{10}{s^3 + 4s^2 + 3s + 1}$

(b) $G(s) = \dfrac{K(s+4)}{s^3 + s^2 + 3s + 9}$

## 2.6 Canonical variables

The canonical-variable, or normal-form, representation of a system often provides a convenient tool for the development of system properties because of the unique decoupled nature of the representation. By decoupled we refer to the fact that in normal form the $n$ first-order equations are completely independent of each other. This decoupling feature also greatly simplifies the determination of the time response of the system.

We begin our discussion once again with the transfer-function representation of the system.

$$\frac{y(s)}{u(s)} = G(s) = \frac{K(c_m s^{m-1} + c_{m-1}s^{m-2} + \cdots + c_2 s + c_1)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)\cdots(s-\lambda_n)} \qquad m \le n \tag{2.6-1}$$

Here, however, the denominator of the transfer function has been written in factored form. The various values of $\lambda_i$ are then the poles of the transfer function or equivalently the zeros of the denominator or *characteristic polynomial*,

$$\begin{aligned}\Delta(s) &= (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)\\ &= s^n + a_n s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_2 s + a_1\end{aligned}$$

For simplicity, the poles are assumed to be distinct throughout this section. Although the normal-form representation may be extended to the case of multiple poles (Zadeh and Desoer, 1963), the added complexity is questionable since there are few practical systems which cannot be satisfactorily approximated by systems with distinct poles (Bellman, 1960). In addition, both the phase-variable and physical-variable representations may be applied directly to systems with multiple poles.

The first step in the development is to make a partial-fraction expansion of $G(s)$

$$\frac{y(s)}{u(s)} = G(s) = \frac{d_1}{s-\lambda_1} + \frac{d_2}{s-\lambda_2} + \frac{d_3}{s-\lambda_3} + \cdots + \frac{d_n}{s-\lambda_n} \tag{2.6-2}$$



This result may also be written as

$$\frac{y(s)}{u(s)} = G(s) = d_1\frac{z_1(s)}{u(s)} + d_2\frac{z_2(s)}{u(s)} + \cdots + d_n\frac{z_n(s)}{u(s)}$$

where

$$\frac{z_i(s)}{u(s)} = \frac{1}{s-\lambda_i} \qquad i = 1, 2, \ldots, n \tag{2.6-3}$$

and

$$y(s) = d_1 z_1(s) + d_2 z_2(s) + \cdots + d_n z_n(s) \tag{2.6-4}$$

In the time domain Eqs. (2.6-3) and (2.6-4) become

$$\begin{aligned}\dot z_i(t) &= \lambda_i z_i(t) + u(t) \qquad i = 1, 2, \ldots, n\\ y(t) &= d_1 z_1(t) + d_2 z_2(t) + \cdots + d_n z_n(t)\end{aligned}$$

Using this result, we may now represent the system in Jordan normal form or more simply just *normal form* as

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}^n u \tag{2.6-5}$$
$$y = \mathbf{c}^{nT}\mathbf{z} \tag{2.6-6}$$

where

$$\mathbf{A} = \begin{bmatrix}\lambda_1 & & & \mathbf{0}\\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n\end{bmatrix} \qquad \mathbf{b}^n = \begin{bmatrix}1\\1\\ \cdots \\1\end{bmatrix} \qquad \text{and} \qquad \mathbf{c}^n = \begin{bmatrix}d_1\\d_2\\ \cdots \\d_n\end{bmatrix}$$

The state variables in this form are often referred to as *canonical variables*, although this term has also been used to describe phase variables (Kalman, 1963). Here the system matrix **A** takes on a particularly simple form as a diagonal matrix of the $\lambda_i$'s. The elements of the $\mathbf{b}^n$ matrix, on the other hand, are all unity, and the elements of $\mathbf{c}^n$ are simply the residues at the respective poles.

A block diagram of the normal-form representation is shown in Fig. 2.6-1. This block diagram emphasizes the simple and decoupled nature of the normal-form representation. For example, the $i$th equation of Eq. (2.6-5) appears as

$$\dot z_i(t) = \lambda_i z_i(t) + u(t)$$

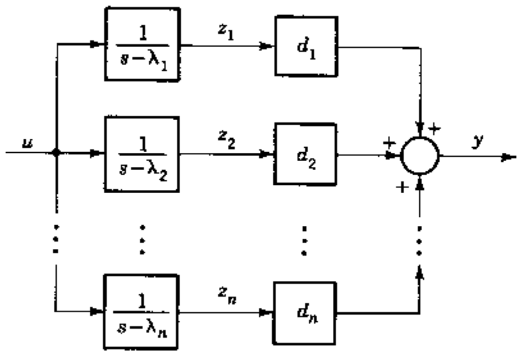


*Fig. 2.6-1* Block diagram for the canonical-variable representation.

This equation may be solved for $z_i(t)$ independently of the other $\mathbf{z}$ coordinates. Not only that, but the solution of the remaining $n - 1$ equations is of the same form as the solution of the $i$th equation. This is an advantage of the normal-form representation.

Figure 2.6-1 also points out the similarity of the normal-form representation to the parallel programming technique of simulating a transfer function on an analog computer. In fact, the close connection between the state-variable representation and the analog-computer simulation of the system is one of the strong points of the modern control methods.

***Example 2.6-1*** Let us represent the system described by the transfer function

$$\frac{y(s)}{u(s)} = G(s) = \frac{2(s+3)}{(s+1)(s+2)}$$

in normal form. Making a partial-fraction expansion of $G(s)$, we obtain

$$G(s) = \frac{4}{s+1} + \frac{-2}{s+2}$$

From this partial-fraction expansion the following relationships may be written directly:

$$\frac{z_1(s)}{u(s)} = \frac{1}{s+1} \qquad \frac{z_2(s)}{u(s)} = \frac{1}{s+2}$$



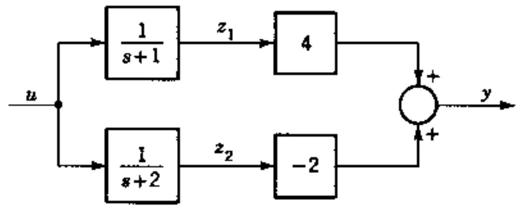


*Fig. 2.6-2* Canonical-variable representation of a second-order system.

and

$$y(s) = 4z_1(s) - 2z_2(s)$$

The normal-form representation is then

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = [4 \quad -2] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Note that this representation could have been written directly from the partial-fraction expansion of $G(s)$. A block-diagram representation of this system is shown in Fig. 2.6-2.

In Sec. 2.4 it is mentioned that the determinant of the matrix $(s\mathbf{I} - \mathbf{A})$ is equal to the denominator polynomial of $G(s)$. By the use of the normal-form representation it is easy to establish this fact. For this case, the matrix $(s\mathbf{I} - \mathbf{A}) = (s\mathbf{I} - \mathbf{\Lambda})$ becomes

$$(s\mathbf{I} - \mathbf{\Lambda}) = \begin{bmatrix} s-\lambda_1 & & & \mathbf{0} \\ & s-\lambda_2 & & \\ & & \cdots & \\ \mathbf{0} & & & s-\lambda_n \end{bmatrix}$$

Then the determinant of $(s\mathbf{I} - \mathbf{\Lambda})$ is

$$\det \begin{bmatrix} s-\lambda_1 & & & \mathbf{0} \\ & s-\lambda_2 & & \\ & & \cdots & \\ \mathbf{0} & & & s-\lambda_n \end{bmatrix} = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n) = \Delta(s)$$

which is exactly the denominator of $G(s)$, as predicted.

In matrix terminology the values of $s$ which satisfy the *characteristic equation* of $\mathbf{A}$, namely,

$$\det(s\mathbf{I} - \mathbf{A}) = 0$$

are referred to as the *eigenvalues* of the matrix $\mathbf{A}$. By the above development, we have shown that, at least in the case of the normal-form representation, the *eigenvalues of* $\mathbf{A}$ *are identical to the poles of* $G(s)$.

The reader has no doubt noticed that little use has been made of matrix manipulations. The vector-matrix notation has served simply as a convenient shorthand way of writing the system equations. For the remaining portion of this section, an alternate approach to the development of the normal form is presented which makes use of many of the matrix methods introduced in Sec. 2.3. This approach is based on a technique known as a *linear transformation of variables*.

As a vehicle for the discussion, consider the general $n$th-order linear system represented in state variables in the usual manner by Eqs. (**Ab**) and (**c**):

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \qquad (\mathbf{Ab})$$
$$y(t) = \mathbf{c}^T\mathbf{x}(t) \qquad (\mathbf{c})$$

Suppose now that a new set of variables $\mathbf{z}$ is introduced, where $\mathbf{x}$ and $\mathbf{z}$ are related by a *nonsingular matrix* $\mathbf{P}$ such that

$$\mathbf{x}(t) = \mathbf{P}\mathbf{z}(t) \qquad (2.6\text{-}7)$$

Here we see that the two sets of variables are *linearly* related by means of the transformation matrix $\mathbf{P}$.

In order to see how the system equations are affected by means of this transformation, we take the derivative of both sides of Eq. (2.6-7) to obtain

$$\dot{\mathbf{x}}(t) = \mathbf{P}\dot{\mathbf{z}}(t) \qquad (2.6\text{-}8)$$

The substitution of Eqs. (2.6-7) and (2.6-8) into Eqs. (**Ab**) and (**c**) yields

$$\mathbf{P}\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{P}\mathbf{z}(t) + \mathbf{b}u(t) \qquad y(t) = \mathbf{c}^T\mathbf{P}\mathbf{z}(t)$$

The first equation can be premultiplied on both sides by $\mathbf{P}^{-1}$ to obtain

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z}(t) + \mathbf{P}^{-1}\mathbf{b}u(t) \qquad y(t) = \mathbf{c}^T\mathbf{P}\mathbf{z}(t)$$



which may be written in the form

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{b}^*u \qquad (2.6\text{-}9)$$
$$y = \mathbf{c}^{*T}\mathbf{z} \qquad (2.6\text{-}10)$$

where

$$\mathbf{A}^* = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \qquad \mathbf{b}^* = \mathbf{P}^{-1}\mathbf{b} \qquad \text{and} \qquad \mathbf{c}^* = \mathbf{P}^T\mathbf{c} \qquad (2.6\text{-}11)$$

The multiplication by $\mathbf{P}^{-1}$ indicates the reason for the initial assumption that $\mathbf{P}$ is nonsingular, since in order to invert $\mathbf{P}$, it is necessary that $\mathbf{P}$ be nonsingular. Note, however, that $\mathbf{P}$ may be *any* nonsingular matrix. Thus a control system initially expressed by one set of state variables in the form (**Ab**) and (**c**) may be converted to an infinite number of alternate representations of the same form.

The linear transformation of the state-variable representations has two interesting and important properties:

(I) $\det(s\mathbf{I} - \mathbf{A}^*) = \det(s\mathbf{I} - \mathbf{A})$
(II) $\mathbf{c}^{*T}(s\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{b}^* = \mathbf{c}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$

In order to establish the first property, we begin by replacing $\mathbf{A}^*$ by $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, so that

$$\det(s\mathbf{I} - \mathbf{A}^*) = \det(s\mathbf{I} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P}) \qquad (2.6\text{-}12)$$

Since $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$, Eq. (2.6-12) may be rewritten

$$\begin{aligned}\det(s\mathbf{I} - \mathbf{A}^*) &= \det(s\mathbf{P}^{-1}\mathbf{P} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P}) \\ &= \det[\mathbf{P}^{-1}(s\mathbf{I} - \mathbf{A})\mathbf{P}]\end{aligned} \qquad (2.6\text{-}13)$$

Since the determinant of a product is the product of the determinants, Eq. (2.6-13) becomes

$$\det(s\mathbf{I} - \mathbf{A}^*) = \det\mathbf{P}^{-1}\det(s\mathbf{I} - \mathbf{A})\det\mathbf{P}$$

The determinant of a matrix is a scalar quantity, and therefore the determinants in the above expression may be rearranged so that

$$\det(s\mathbf{I} - \mathbf{A}^*) = \det\mathbf{P}^{-1}\det\mathbf{P}\det(s\mathbf{I} - \mathbf{A}) \qquad (2.6\text{-}14)$$

Again making use of the fact that the product of determinants is the determinant of the product, we rewrite Eq. (2.6-14) as

$$\begin{aligned}\det(s\mathbf{I} - \mathbf{A}^*) &= \det(\mathbf{P}^{-1}\mathbf{P})\det(s\mathbf{I} - \mathbf{A}) \\ &= \det(s\mathbf{I} - \mathbf{A})\end{aligned} \tag{2.6-15}$$

which is the result we wished to establish.

This property allows us to conclude that *the* $\det(s\mathbf{I} - \mathbf{A})$ *and therefore the eigenvalues of the matrix* $\mathbf{A}$ *are invariant under any linear transformation* of variables. Since we have established in the case of canonic variables that $\det(s\mathbf{I} - \mathbf{A}) = \Delta(s)$ and that the eigenvalues of $\mathbf{A}$ are equal to the poles of $G(s)$, these same properties must be true, as they are invariant, for any state-variable representation that may be found by a linear transformation of the normal form. But this is, in fact, every state-variable representation, and therefore we draw the following conclusions: for *any system representation of the form* ($\mathbf{Ab}$) *and* ($\mathbf{c}$), *the*

$$\det(s\mathbf{I} - \mathbf{A}) = \Delta(s)$$

*and therefore the eigenvalues of* $\mathbf{A}$ *are the poles of* $G(s)$. Since the eigenvalues of $\mathbf{A}$ are independent of which $\mathbf{A}$ matrix we use to represent the system, one usually speaks of the *eigenvalues of the system*, since they are an intrinsic property of the system and independent of the representation used.

It should be noted that although we have established the fact that

$$\det(s\mathbf{I} - \mathbf{A}^*) = \det(s\mathbf{I} - \mathbf{A})$$

this does *not* imply that

$$(s\mathbf{I} - \mathbf{A}^*) = (s\mathbf{I} - \mathbf{A}) \tag{2.6-16}$$

In fact, if Eq. (2.6-16) were true, it would not be possible to prove property II, which is

$$\text{(II)} \quad \mathbf{c}^{*T}(s\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{b}^* = \mathbf{c}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$$

In order to demonstrate this second property, we begin once again by substituting the definitions of Eqs. (2.6-11) into the expression to obtain

$$\mathbf{c}^{*T}(s\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{b}^* = \mathbf{c}^T\mathbf{P}(s\mathbf{I} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P})^{-1}\mathbf{P}^{-1}\mathbf{b}$$



Taking the $\mathbf{P}$ and $\mathbf{P}^{-1}$ inside the inverse, we obtain

$$\mathbf{c}^{*T}(s\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{b}^* = \mathbf{c}^T(s\mathbf{P}\mathbf{P}^{-1} - \mathbf{A})^{-1}\mathbf{b} = \mathbf{c}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$$

which is the desired result.

Since $\mathbf{c}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = G(s)$, this property is nothing more than a statement that the transfer function associated with a state-variable representation is invariant under a linear transformation. This conclusion is simply a restatement of the uniqueness property of the transfer function.

Having investigated some of the properties of linear transformations, let us see how this technique can be used to develop the canonical-variable representation of systems. In particular, we wish to find a matrix $\mathbf{P}$ which transforms a general state-variable representation, ($\mathbf{Ab}$) and ($\mathbf{c}$), into the normal-form representation

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}^n u \tag{2.6-5}$$
$$y = \mathbf{c}^{nT}\mathbf{z} \tag{2.6-6}$$

In terms of the transformation relations of Eqs. (2.6-11) this requirement is equivalent to the requirement that $\mathbf{P}$ satisfy the following equations:

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{A} \tag{2.6-17}$$
$$\mathbf{P}^{-1}\mathbf{b} = \mathbf{b}^n \tag{2.6-18}$$
$$\mathbf{P}^T\mathbf{c} = \mathbf{c}^n \tag{2.6-19}$$

Since the elements of $\mathbf{A}$ are the poles of the transfer function $G(s)$ or, equivalently, the eigenvalues of $\mathbf{A}$, if $G(s)$ is known, the $\lambda_i$'s and therefore $\mathbf{A}$ are known. The only unknown in the first two equations then is the matrix $\mathbf{P}$. In order to put these equations in a more convenient form, premultiply both sides of the equations by $\mathbf{P}$ to obtain

$$\mathbf{A}\mathbf{P} = \mathbf{P}\mathbf{A} \tag{2.6-20}$$

and

$$\mathbf{b} = \mathbf{P}\mathbf{b}^n \tag{2.6-21}$$

These two matrix equations generate a set of $n^2 + n$ linear equations in the elements of $\mathbf{P}$ which may be solved to determine $\mathbf{P}$.

Since there are only $n^2$ elements in $\mathbf{P}$, the reader may wonder how it is possible to satisfy $n^2 + n$ equations. The fact is that if the system is controllable, only $n^2$ of these $n^2 + n$ equations are linearly independent, and therefore the equations may always be satisfied. The reason that the system must be controllable can be understood if one considers the normal form. In the normal-form representation of Eqs. (2.6-5) and (2.6-6), it is easy to see that the system must be controllable since each state variable is decoupled and individually affected by the control $u$. Therefore in order to transform a system into this normal-form representation, it is necessary that the system be controllable.

By a similar argument it is possible to show that the system is observable if all the elements of $\mathbf{c}^n$, namely, $d_1, d_2, \ldots, d_n$, are nonzero. In fact, some authors (Gilbert, 1963) have used this approach as a definition of controllability and observability. This is done by taking any $\mathbf{P}$ matrix which satisfies Eq. (2.6-20) and then examining the resulting $\mathbf{b}^n$ and $\mathbf{c}^n$ vectors. If $\mathbf{b}^n$ and $\mathbf{c}^n$ contain only nonzero elements, the system is said to be controllable and observable.[1]

Since we have assumed that all the systems we shall deal with are controllable and observable, the requirement for controllability offers no problem. It is therefore always possible to solve Eqs. (2.6-20) and (2.6-21) to determine $\mathbf{P}$. The observability property may also be used to check the calculations by examining whether all the $d_i$'s are nonzero, as they should be.

***Example 2.6-2*** In order to illustrate the above procedure for determining $\mathbf{P}$, let us consider once again the system of Example 2.6-1. Here we assume that the system is initially represented in phase-variable form rather than as a transfer function, so that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$$y = [3 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Although the eigenvalues of the system, the poles of $G(s)$, are known, let us assume that they are unknown and calculate them from the characteristic equation of $\mathbf{A}$, which is

$$\det\,(s\mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} = s^2 + 3s + 2 = 0$$

[1] The reader is reminded that we have assumed that the eigenvalues are distinct. If this requirement is not satisfied, these statements must be modified.



The values of $s$ which satisfy this equation are

$$s = -1 \qquad \text{and} \qquad s = -2$$

and therefore

$$\lambda_1 = -1 \qquad \text{and} \qquad \lambda_2 = -2$$

so that $\mathbf{A}$ becomes

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

The matrix $\mathbf{P}$ may now be found by the use of Eqs. (2.6-20) and (2.6-21), which are in this case

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

and
$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

If these equations are expanded and the elements of the resulting matrices equated, the following six simultaneous equations result.

$$\begin{aligned} p_{21} &= -p_{11} & -2p_{11} - 3p_{21} &= -p_{21} \\ p_{22} &= -2p_{12} & -2p_{12} - 3p_{22} &= -2p_{22} \\ 0 &= p_{11} + p_{12} & 2 &= p_{21} + p_{22} \end{aligned}$$

The reader will immediately recognize that the first and second equations and the third and fourth equations are identical, and therefore there are only the four following equations to satisfy:

$$\begin{aligned} p_{21} &= -p_{11} & p_{22} &= -2p_{12} \\ 0 &= p_{11} + p_{12} & 2 &= p_{21} + p_{22} \end{aligned}$$

Simultaneously solving these four equations gives the following $\mathbf{P}$ matrix:

$$\mathbf{P} = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$$

It is suggested that the reader verify that **P** satisfies Eqs. (2.6-17) and (2.6-18). We may now use Eq. (2.6-19) to determine $\mathbf{c}^n$.

$$\mathbf{P}^T\mathbf{c} = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \mathbf{c}^n = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

which is identical to the answer obtained in Example 2.6-1.

The reader may reasonably question why this elaborate linear-transformation approach to the canonical-variable representation has been developed, since it is possible to achieve the same result in an easier manner directly from the transfer function. There are several answers to this question. First, the linear-transformation technique is valuable in modern control concepts since it provides a procedure for relating two different sets of state variables. Second, the transfer function for a system may not be known if one has initially represented the system in state-variable form. Therefore rather than obtaining the transfer function and then representing the system in normal form, the linear-transformation technique allows us to transform the system to normal form directly. Third, it is often desirable to know the transformation matrix **P** even if the transfer function is known. This is the case whenever we wish to translate results obtained in normal form into another set of state variables.

Although the normal-form representation is valuable because of the simple decoupled nature of the resulting first-order equations, it does have two disadvantages. First, the normal form cannot be determined by inspection from the transfer function as phase variables can. This means that it is more difficult to relate the normal form and transfer functions than to relate phase variables and transfer functions.

Second, the canonical variables, like phase variables, are not real physical variables. This fact is perhaps most strikingly obvious if there are complex eigenvalues. In this case some of the canonical variables are also complex.

This means that **x** and **z** are related by a **P** matrix, some of the elements of which are complex. Therefore, in the physical system, or alternately on the block-diagram representation of the physical system, there is no set of real state variables that can be combined to form the canonical variables.

**Exercises 2.6** *2.6-1*. For the systems shown below, express the equations of motion in (1) phase variables and (2) normal coordinates. Find the normal coordinates by linear transformation from the phase-variable representation and directly from the transfer function.

(*a*) $G(s) = \dfrac{(s+8)}{(s+2)(s+6)}$ (*b*) $G(s) = \dfrac{10(s+4)}{s^3 + 3s^2 + 2s}$

*answers:*

(*a*) $\dot{\mathbf{z}} = \begin{bmatrix} -2 & 0 \\ 0 & -6 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\mathbf{u} \qquad y = [\tfrac{3}{2} \quad -\tfrac{1}{2}]\mathbf{z}$

(*b*) $\dot{\mathbf{z}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\mathbf{u} \qquad y = [20 \quad -30 \quad 10]\mathbf{z}$

*2.6-2*. The state-variable equations for a particular system are given below. Describe this system in canonical variables using the linear-transformation techniques. Draw a block diagram corresponding to the given equations and one corresponding to the canonical-form representation.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}u$$

$$y = [1 \quad 0]\mathbf{x}$$

*answer:*

$$\dot{\mathbf{z}} = \begin{bmatrix} -1-j & 0 \\ 0 & -1+j \end{bmatrix}\mathbf{z} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u$$

$$y = [2j \quad -2j]\mathbf{z}$$

## 2.7 *Physical variables*

The method of system representation in real physical-system variables is much more intuitive than the methods of the previous two sections. In fact, the reader may feel that this approach is so straightforward that it should not be called a method! The physical-system-variable representa-

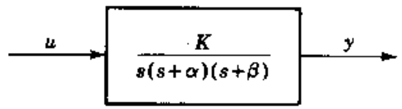


*Fig. 2.7-1* Block diagram that can be represented by several state-variable forms.

tion is discussed separately here as a first step in emphasizing an outlook that is advocated throughout this book.

By way of introduction, let us start with the block-diagram approach used in the discussion of phase and canonic variables. Consider, for example, the rather specific block diagram of Fig. 2.7-1. One may think of this block diagram as originating from a simple position-control system containing a motor with electrical and mechanical time constants. This is not a unique interpretation by any means, as the block diagram may describe any other third-order system. In fact, that is the point. Because an overall block diagram is all that is given, one can only speculate about the physical origins of the example.

In order to realize several state-variable representations, the block diagram of Fig. 2.7-1 can be redrawn, as in Fig. 2.7-2$a$, $b$, or $c$. If this is done, as shown, so that each block contains only one pole, the state variables can be identified, and state-variable equations can be written directly. For Fig. 2.7-2$b$, for instance, the state-variable equations are

$$
\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= -\beta x_2 + \tfrac{1}{2}x_3\\
\dot{x}_3 &= -\alpha x_3 + 2Ku\\
y &= x_1
\end{aligned}
$$

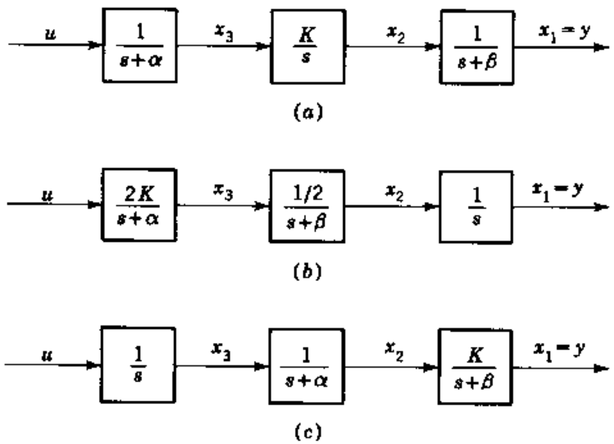


*Fig. 2.7-2* Several sets of state variables from the same transfer function.



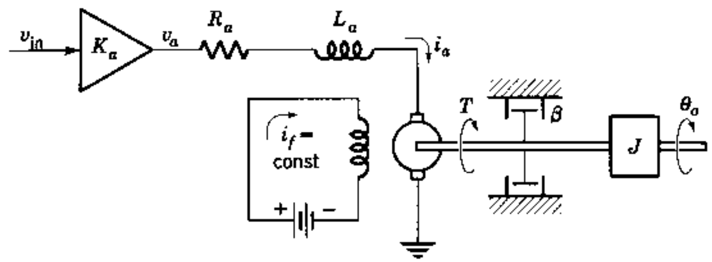


*Fig. 2.7-3* Open-loop positioning system.

It is not possible to say whether or not these particular state variables have any physical meaning.

The obvious approach is to break the block diagram up in such a way that the physical-system variables can be identified. A more basic starting point is the governing equations themselves.

Let us assume that the block diagram of Fig. 2.7-1 actually does represent an open-loop positioning system, shown diagrammatically in Fig. 2.7-3. On the diagram physical-system variables and parameters are identified, where

$\theta_o$ = output position angle
$v_{in}$ = input voltage
$v_a$ = output voltage of linear amplifier
$i_a$ = motor armature current
$i_f$ = motor field current, assumed constant
$K_a$ = gain of linear amplifier, assumed to have no significant time constants
$R_a$ = resistance of armature winding
$L_a$ = inductance of armature winding
$J$ = inertial load
$\beta$ = viscous-damping constant
$K_T$ = torque constant of motor
$K_v$ = back-emf constant of motor

The differential equations that govern the dynamics of the system are

$$J\ddot{\theta}_o + \beta\dot{\theta}_o = K_T i_a \qquad (2.7\text{-}1)$$
$$L_a\dot{i}_a + R_a i_a = V_a - K_v\dot{\theta}_o \qquad (2.7\text{-}2)$$

On the basis of these equations, the block diagram for this system can be drawn, as Fig. 2.7-4.

With all the system variables identified, it is a simple matter to choose physically meaningful state variables. The output $\theta_o$ is chosen as

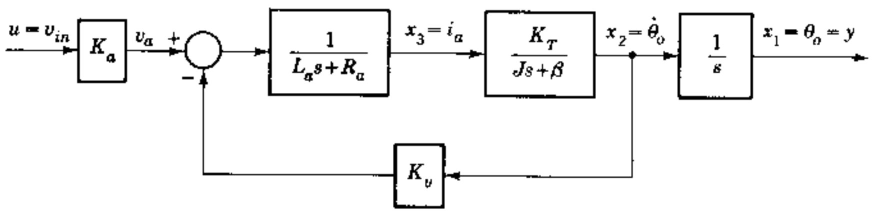


*Fig. 2.7-4* Block-diagram representation of the control system of Fig. 2.7-3.

$x_1$ so that $y = x_1$, and the other choices are

$$x_2 = \dot{\theta}_o \quad . \quad x_3 = i_a \qquad u = v_{in} \tag{2.7-3}$$

In terms of the state variables defined in Eq. (2.7-3) the state equations may be written directly from Eqs. (2.7-1) and (2.7-2) or from Fig. 2.7-4. First, from the definitions of $x_1$ and $x_2$, we have

$$\dot{x}_1 = x_2$$

Next, having substituted the definitions into Eq. (2.7-1), we obtain

$$J\ddot{x}_1 + \beta\dot{x}_1 = K_T x_3$$

Since $\dot{x}_1 = x_2$, this may be written as

$$J\dot{x}_2 + \beta x_2 = K_T x_3$$

or

$$\dot{x}_2 = -\frac{\beta}{J}x_2 + \frac{K_T}{J}x_3$$

Finally we use Eq. (2.7-2) to write

$$\dot{x}_3 = -\frac{R_a}{L_a}x_3 - \frac{K_v}{L_a}x_2 + \frac{K_a}{L_a}u$$

If these results are collected, the following physical-system representation is obtained:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{\beta}{J}x_2 + \frac{K_T}{J}x_3$$

$$\dot{x}_3 = -\frac{K_v}{L_a}x_2 - \frac{R_a}{L_a}x_3 + \frac{K_a}{L_a}u \tag{2.7-4}$$

$$y = x_1 \tag{2.7-5}$$



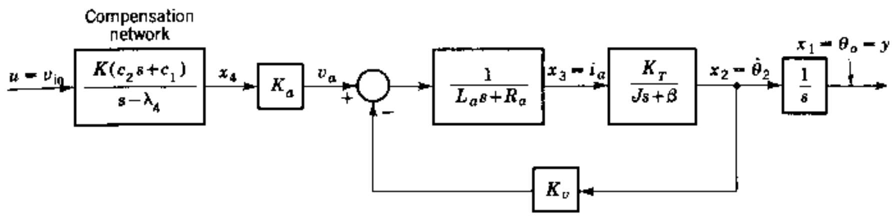


*Fig. 2.7-5* Block diagram of Fig. 2.7-4 with a compensation network added.

These are the describing equations for the given positioning system that is pictured in Fig. 2.7-3 or equivalently represented by the block diagram of Fig. 2.7-4. Of course, Fig. 2.7-4 could be reduced to the overall system of Fig. 2.7-1. Obviously, physically meaningful state variables could not be chosen from that diagram.

One problem arises in the use of physical variables that does not occur in the other methods of system representation. It is introduced by zeros in $G(s)$. Each zero of $G(s)$ is always intimately related to a specific pole of $G(s)$. For example, if $G(s)$ includes a network, e.g., a lead or lag network, each of these networks introduces a zero. Physically this zero is associated with the pole of the network. If a lead or lag network is included in the position system that has been discussed in this section, the block diagram appears as in Fig. 2.7-5. Here the output of the compensation network is a physical variable, e.g., a voltage, and the relationship between $u$ and $x_4$ is specified by the interconnecting block. In the time domain, the fourth state equation is

$$\dot{x}_4 = \lambda_4 x_4 + Kc_2\dot{u} + Kc_1 u$$

Since $\dot{u}$ is involved on the right-hand side, this equation is not compatible with the form (**Ab**).

This problem was encountered before in the discussion of phase variables. There the means used to avoid the difficulty were a modification of either the output vector **c** or the control vector **b** (see Sec. 2.5). We have no such freedom in this case, since both the output and the control are unique and are defined by the physical system under study. If we wish to retain the use of real physical variables, we are not free to define alternate control or output expressions, no matter how convenient it might be. We do not meet this problem again until Chap. 9.

It has been implied throughout this chapter that the governing system differential equations and the block diagram convey the same information. They do, but only in an overall input-to-output sense. This example clearly demonstrates that a variety of system representations can be chosen from a block diagram, although many of these representations are not composed of physically real variables.

A better way to treat the topic of system representation is to ignore the block-diagram approach completely and simply write the state equations from the governing differential equations. That is the approach of Chap. 3, where use is made of the state function of Lagrange and Lagrange's equations are used to determine the state equations.

In this and the preceding two sections, we have discussed three different techniques for obtaining the state-variable representation of a system initially described by an overall input-output transfer function. These three methods are not the only means of accomplishing the desired result; in fact, there is an infinite variety of means, but the methods described here are the most common and effective. Let us summarize those results here for convenient reference.

First, phase variables afford a simple and direct means of translating the transfer-function information into a unique state-variable form. At the same time, they lack physical significance, particularly if the system has zeros, and they also offer little assistance in obtaining the time response.

Canonical variables, on the other hand, offer an indirect means of changing the transfer function into a unique state-variable representation. At the same time, however, the resulting system representation is in a simple decoupled form which facilitates an investigation of system properties and a determination of the time response. Once again, the state variables are not real physical variables and may, in fact, be complex quantities.

The use of the physical-system-variable representation, by its very essence, implies that the resulting state variables are real physical variables which can be measured and used for control purposes. By the very fact that this means of system representation is intimately related to the physical system, the approach no longer produces a unique form for the resulting state-variable representation. The same transfer function, for example, may generate two different representations if the physical system is different. The effort involved in determining this state-variable representation and its time response probably falls somewhere between the limits of the phase-variable and canonical-variable approaches.

Even though we reject the use of the phase variables and the canonical variables on physical grounds, this does not mean that we shall no longer have occasion to consider these particular coordinate frames. Because the mathematical characteristics of these sets of coordinates are so desirable, we shall continue to use them, as long as we realize that in the physical system we must ultimately deal with physical variables.



**Exercises 2.7** *2.7-1.* For the system shown below find the equations of motion in the form (**Ab**) and (**c**). Use two different sets of variables, determined by breaking $G(s)$ into individual blocks, as indicated.

(*a*) $G(s) = \dfrac{2}{s+3}\dfrac{s+2}{s+1}\dfrac{1}{s}$ (*b*) $G(s) = \dfrac{2}{s}\dfrac{1}{s+1}\dfrac{s+2}{s+3}$

*answers:*

(*a*) $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}u \qquad y = x_1$

(*b*) $\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}u \qquad y = x_1$

*2.7-2.* Find a block-diagram representation for the system shown in Fig. 2.7-6. Express the equations of motion in the form (**Ab**) and (**c**), with $x_1 = \theta_o$; $x_2 = e_t$, the tachometer voltage; and $x_3 = i_f$, the field current.

*answer:*

$$\dot{x}_1 = \frac{1}{K_T}x_2 \qquad \dot{x}_2 = -\frac{\beta}{J}x_2 + \frac{K_eK_T}{J}x_3$$

$$\dot{x}_3 = -\frac{1}{L_f}x_2 - \frac{R_f}{L_f}x_3 + \frac{K_a}{L_f}u$$

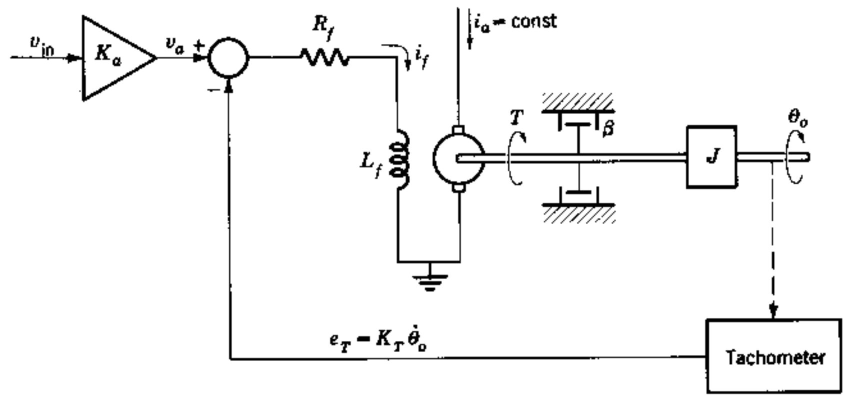


***Fig. 2.7-6*** Exercise 2.7-2.

## 2.8 Representation of nonlinear systems

An advantage of time-domain techniques is that they can be directly extended to include time-varying, sampled-data, nonlinear systems, or a combination of both. As an illustration of this feature, this section outlines how nonlinear-gain-type control systems can be represented in state-variable form.

Consider the block diagram of Fig. 2.8-1. This may be thought of as a continuation of the example of the positioning system of Sec. 2.7. Here, however, the loop has been closed by assuming that the input $u$ is equal to the negative of the output, $y = x_1$. A further complication is that the amplifier is no longer assumed to be linear but is allowed to saturate, as indicated in Fig. 2.8-1. The output of the amplifier is therefore

$$v_a = f(u) = f(-y) = f(-x_1)$$

Making the substitution of this nonlinear $v_a$ for the quantity $K_a u$ in Eq. (2.7-4), we obtain the following state-variable representation of this nonlinear closed-loop position regulator:

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -\frac{\beta}{J}x_2 + \frac{K_T}{J}x_3$$

$$\dot{x}_3 = -\frac{K_v}{L_a}x_2 - \frac{R_a}{L_a}x_3 + \frac{1}{L_a}f(-x_1) \qquad y = x_1$$

The phase-variable and canonical-variable representation may also be used for nonlinear systems. Consider, for example, the general $n$th-order nonlinear-gain closed-loop system shown in Fig. 2.8-2. The phase-

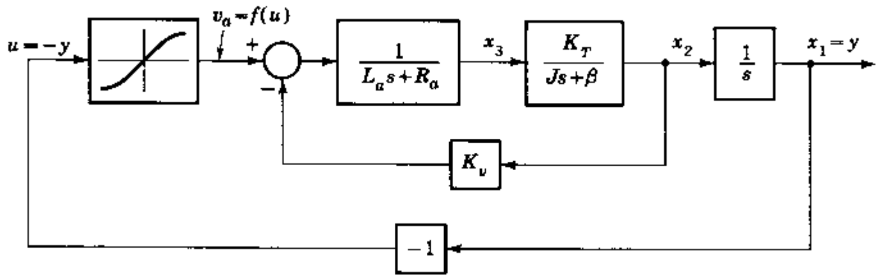


*Fig. 2.8-1* Positioning system of Sec. 2.7 with the loop closed and a nonlinear amplifier.



variable description of this system is

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\cdots\cdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= -a_1x_1 - a_2x_2 - \cdots - a_nx_n + Kf(-y)
\end{aligned}$$

where

$$y = [c_1 \quad c_2 \quad \cdots \quad c_m \quad 0 \quad \cdots \quad 0]\mathbf{x}$$

Here $G(s)$ is given by

$$\frac{y(s)}{u(s)} = G(s) = \frac{K(c_m s^{m-1} + c_{m-1}s^{m-2} + \cdots + c_2 s + c_1)}{s^n + a_n s^{n-1} + \cdots + a_2 s + a_1}$$

In order to represent this nonlinear system in canonical variables, we begin, as before, by making a partial-fraction expansion of the transfer function of the linear portion of the system.

$$G(s) = \frac{d_1}{s-\lambda_1} + \frac{d_2}{s-\lambda_2} + \cdots + \frac{d_n}{s-\lambda_n}$$

When the same definition for the canonical variables as in Sec. 2.6 is used, the canonical-variable representation of the nonlinear-gain system is

$$\begin{aligned}
\dot{z}_1 &= \lambda_1 z_1 + f(-y) \\
\dot{z}_2 &= \lambda_2 z_2 + f(-y) \\
&\cdots\cdots\cdots\cdots \\
\dot{z}_n &= \lambda_n z_n + f(-y)
\end{aligned} \qquad (2.8\text{-}1)$$

where

$$y = [d_1 \quad d_2 \quad \cdots \quad d_n]\mathbf{z}$$

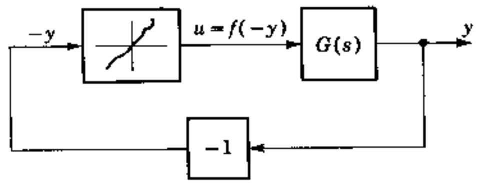


*Fig. 2.8-2* General $n$th-order nonlinear-gain closed-loop system.

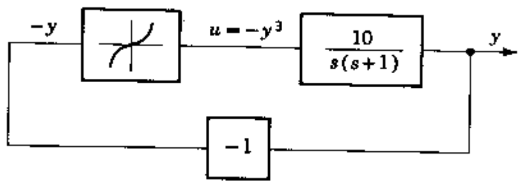


*Fig. 2.8-3* Exercise 2.8-1.

This is a familiar form, but now $u$ in each of the $n$ equations has been replaced by $f(-y)$. Since $y$ is actually a linear combination of all the canonical variables, we see that in the nonlinear case the equations have not been decoupled. Nonlinear-gain-type representations as in Eqs. (2.8-1) are referred to as being in *Lurie's canonical form.*

In this section, the state-variable representations of the preceding sections have been applied to the description of nonlinear-gain systems. This ability of the state-variable representation of systems to encompass nonlinear as well as time-varying systems is one of the strongest recommendations of the modern method.

In order to represent time-varying systems, it is only necessary to allow the matrices **A**, **B**, and **C** to become functions of time as

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \qquad \mathbf{y} = \mathbf{C}(t)\mathbf{x}(t)$$

**Exercises 2.8** *2.8-1.* Represent the nonlinear-gain system shown in Fig. 2.8-3 in (*a*) phase-variable form and (*b*) normal form.

*answer:*

(*a*) $$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 10 \end{bmatrix}(-x_1)^3 \qquad y = x_1$$

(*b*) $$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}(-10z_1 + 10z_2)^3$$

$$y = [10 \quad -10]\mathbf{z}$$

## 2.9 Summary and conclusion

This chapter has dealt with one aspect of problem formulation, namely, system representation in state-variable form. The discussion has been based almost entirely upon the assumption that the control system being studied is known in the form of a block diagram. Admittedly, this is not a valid assumption in many cases. This procedure has been adopted purely as a means of transition from the familiar block-diagram representation in the frequency domain to the vector-matrix notation and first-order differential equations of the time domain.

Although the general linear transformation of variables has been discussed briefly, this chapter has emphasized three particular means of system description: description in phase variables, canonical variables, and physical-system variables. The phase variables prove to be the simplest to realize, while the canonical variables are the simplest to solve. Either of these coordinate frames may have particular state variables that are not measurable or real. The use of physical-system variables has been advocated, with a view toward the future requirement of feeding back all the state variables.

The following chapter presents a more general approach to the question of system representation in terms of physical-system variables by the use of the state function of Lagrange.



## 2.10 Problems

*2.10-1.* Show that $\mathbf{A}$ adj $\mathbf{A} = \det(\mathbf{A})\mathbf{I}$ and verify for

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & -3 \\ 0 & 0 & 2 \\ -2 & 0 & -2 \end{bmatrix}$$

*2.10-2.* Show that $[(\mathbf{AB})^{-1}]^T = (\mathbf{A}^T)^{-1}(\mathbf{B}^T)^{-1}$ and verify for

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$$

*2.10-3.* Find the system representation (**AB**) and (**c**) for the block diagram of Fig. 2.10-1 using the variables indicated on the diagram. Discuss the controllability and observability of the system.

*2.10-4.* In the Ward-Leonard system shown in Fig. 2.10-2 the generator is driven at a constant speed, and the motor is excited by a constant current. The load torque $T_L$ and the generator field voltage $v_f$ are the system inputs, while the motor-shaft position $\theta_m$ is the output. Make assumptions that result in linear differential equations and draw a block diagram for the system showing $\theta_m$, $\dot{\theta}_m$, $e_g$, $v_f$, and $T_L$. Express the system equations in the form (**AB**) and (**c**) using the above variables.

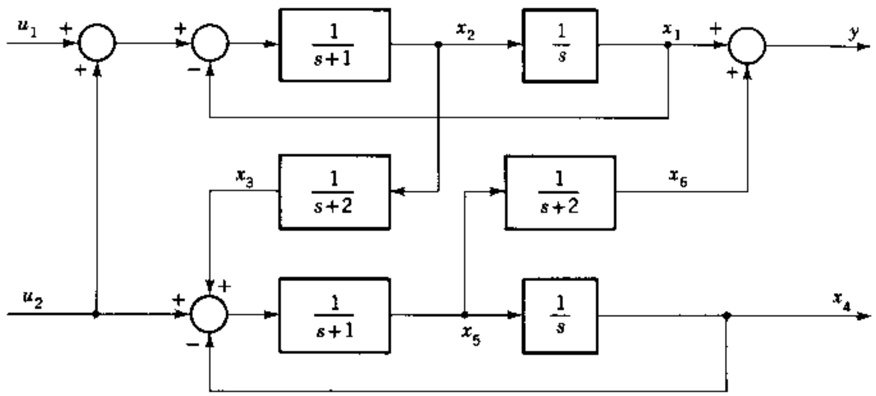


*Fig. 2.10-1* Prob. 2.10-3.

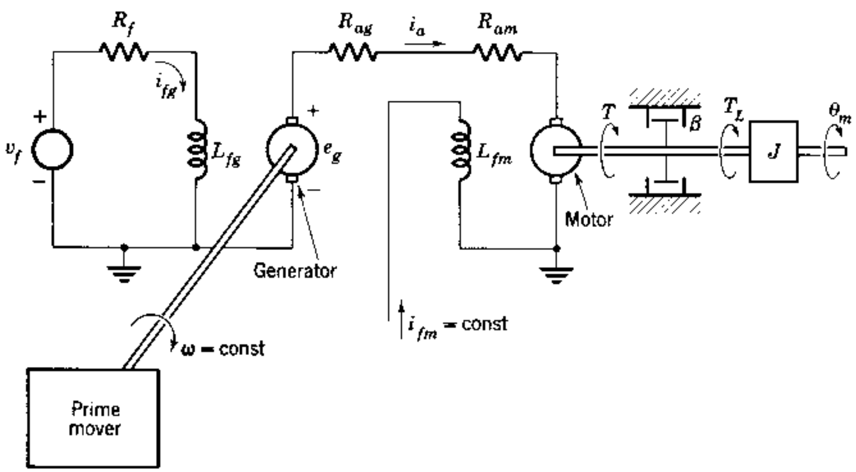


*Fig. 2.10-2* Prob. 2.10-4.

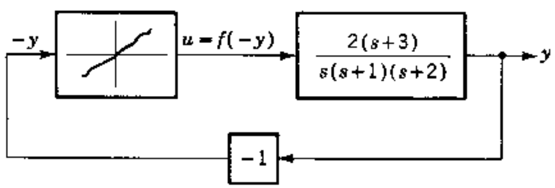


*Fig. 2.10-3* Prob. 2.10-5.



*2.10-5.* Express the equations of motion in normal form for the nonlinear system shown in Fig. 2.10-3.

*2.10-6.* For the system given below represent the system in (*a*) phase variables, (*b*) canonical variables, and (*c*) physical variables, dividing the transfer function as shown. Find the transformation matrices relating each representation to the others. Verify in all cases that the transfer function is unique.

$$G(s) = \frac{10}{s+2}\frac{s+5}{s+1}\frac{1}{s}$$

*2.10-7.* Find the transfer function for the system given below (*a*) by means of Eq. (2.4-11) and (*b*) by block-diagram manipulations.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & -1 \\ -1 & -2 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 1]\mathbf{x}$$

*2.10-8.* For the system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 1]\mathbf{x}$$

use the linear transformation

$$\mathbf{z} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\mathbf{x}$$

to find the system representation in terms of the $\mathbf{z}$ variables. Verify that

(*a*) $\det(s\mathbf{I} - \mathbf{A}^*) = \det(s\mathbf{I} - \mathbf{A})$
(*b*) $\mathbf{c}^{*T}(s\mathbf{I} - \mathbf{A}^*)\mathbf{b}^* = \mathbf{c}^T(s\mathbf{I} - \mathbf{A})\mathbf{b}$

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# *three* *system representation and the state function of lagrange*

## *3.1 Introduction and outline of chapter*

The previous chapter served to demonstrate a variety of means by which a conventional control system can be described in state-variable form. Each of the methods discussed for achieving such representation assumed that the given system was described in transfer-function form. The resulting variables may or may not have any physical meaning, although it was pointed out that physical variables can be chosen from the block diagram if care is exercised in their choice. If physical variables are to be chosen, however, more knowledge of the system must be available than just the transfer function.