⑥

1 (a) $\lambda_1 = -i \qquad \lambda_2 = i \qquad \lambda_3 = -1$

$$\underline{e}_1 = \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix} \quad \underline{e}_2 = \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix} \quad \underline{e}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

(b) $\underline{\Phi} = E e^{\Lambda t} E^{-1}$

$$\underline{E} = \begin{bmatrix} 1 & 1 & 1 \\ -i & i & -1 \\ 1 & 1 & 1 \end{bmatrix} \quad e^{\Lambda t} = \begin{bmatrix} e^{-it} & 0 & 0 \\ 0 & e^{it} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

$$E^{-1} = \begin{bmatrix} \frac{i+1}{4} & \frac{i}{2} & \frac{i-1}{4} \\ \frac{1-i}{4} & -\frac{i}{2} & \frac{-i-1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

(c) $$E^{-1}B = \begin{bmatrix} (i+1)a-2i & \frac{(i+1)b+2i}{2} \\ \frac{(1-i)a+2i}{2} & -\frac{(i-1)b+2i}{2} \\ 2a & b \end{bmatrix} \Rightarrow$$ controllable if a and $b \neq 0$

$$CE = \begin{bmatrix} \alpha - i & \alpha + i & \alpha - 1 \\ 2 - i\beta & i\beta + 2 & 2 - \beta \end{bmatrix}$$ obs if $\alpha \neq 1$ & $\beta \neq 2$

$$\underline{\Phi} = \frac{1}{2} \begin{bmatrix} \sin(t) + \cos(t) + e^{-t} & 2\sin(t) & \sin(t) - \cos(t) + e^{-t} \\ -\sin(t) + \cos(t) - e^{-t} & 2\cos(t) & \sin(t) + \cos(t) - e^{-t} \\ -\sin(t) - \cos(t) + e^{-t} & -\sin(t)2 & -\sin(t) + \cos(t) + e^{-t} \end{bmatrix}$$

2.

(a) $$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -200 & -230 & -31 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

$$y = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix} \underline{x}$$

(b)

$$\Phi = \begin{bmatrix} 1 & 1 & 1 \\ -10 & -1 & -20 \\ 100 & 1 & 400 \end{bmatrix} \begin{bmatrix} e^{-10t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-20t} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -10 & -1 & -20 \\ 100 & 1 & 400 \end{bmatrix}^{-1}$$

(c)

$$y(t) = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix} \Phi \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix} E \, e^{\Lambda t} E^{-1} \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 3 & -16 \end{bmatrix} e^{\Lambda t} \begin{bmatrix} -\frac{1}{9} \\ \frac{10}{171} \\ \frac{1}{19} \end{bmatrix} = \begin{bmatrix} -6 & 3 & -16 \end{bmatrix} \begin{bmatrix} -\frac{e^{-10t}}{9} \\ \frac{10e^{-t}}{171} \\ \frac{e^{-20t}}{19} \end{bmatrix}$$

$$y(t) = \frac{6}{9} e^{-10t} + \frac{30}{171} e^{-t} - \frac{16}{19} e^{-20t}$$

3

(a) $H = \frac{1}{2}\left[x^2 + u^2\right] + \lambda\left[10x + 4u\right]$

(b) $\frac{\partial H}{\partial u} = 0$ $\qquad \boxed{u^\circ = -\lambda 4}$

$$H^\circ = \frac{1}{2}\left[x^2 + 16\lambda^2\right] + \lambda\left[10x - 16\lambda\right]$$

(c)

$$\dot{x} = 10x - 16\lambda$$
$$-\dot{\lambda} = x + 10\lambda$$

(d)

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} 10 & -16 \\ -1 & 10 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \qquad \begin{matrix} x(0) = 1 \\ x(1) = 2 \end{matrix}$$

$\alpha_1 = -2p \qquad \alpha_2 = 2p \qquad$ where $\boxed{p = \sqrt{24}}$

$\underline{e}_1 = \begin{bmatrix} 1 \\ \frac{p+5}{8} \end{bmatrix} \qquad \underline{e}_2 = \begin{bmatrix} 1 \\ \frac{-p+5}{8} \end{bmatrix}$

$$\underline{\Phi} = \frac{1}{16p} \begin{bmatrix} 8 & 8 \\ p+5 & -p+5 \end{bmatrix} \begin{bmatrix} e^{-2pt} & 0 \\ 0 & e^{2pt} \end{bmatrix} \begin{bmatrix} p-5 & 8 \\ p+5 & -8 \end{bmatrix}$$

$$\underline{\Phi} = \frac{1}{16p} \begin{bmatrix} 8(p+5)e^{2pt} + 8(p-5)e^{-2pt} & 64(e^{-2pt} - e^{2pt}) \\ (5-p)(p+5)e^{2pt} + (p-5)(p+5)e^{-2tp} & 8(p+5)e^{-2pt} - 8(5-p)e^{2pt} \end{bmatrix}$$

@ $t=2$

$$2 = \frac{1}{16p}\left[8(p+5)e^{2p} + 8(p-5)e^{-2p} + 64\lambda(0)(e^{-2p} - e^{2p})\right]$$

$$\boxed{\lambda(0) = \frac{32p - \{8(p+5)e^{2p} + 8(p-5)e^{-2p}\}}{64(e^{-2p} - e^{2p})}}$$

$$u^{\circ} = -4 \Big\{ x(0)\Big[(5-p)(p+5)e^{2pt} + (p-5)(p+5)e^{-2pt}\Big]$$

$$+ \lambda(0)\Big[8(p+5)e^{-2pt} - 8(5-p)e^{2pt}\Big] \Big\}$$

where $x(0) = 1$