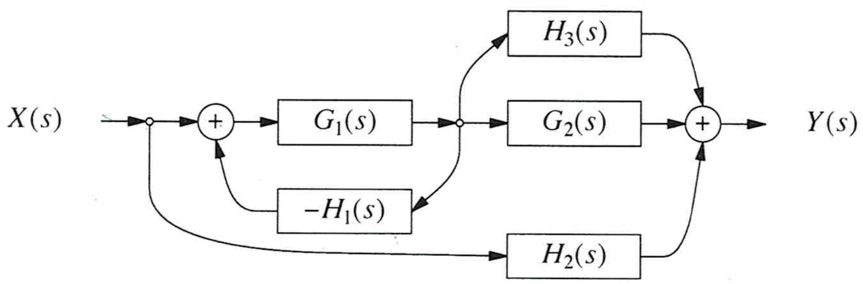
Department of Electrical and Computer Engineering
University of Massachusetts Lowell

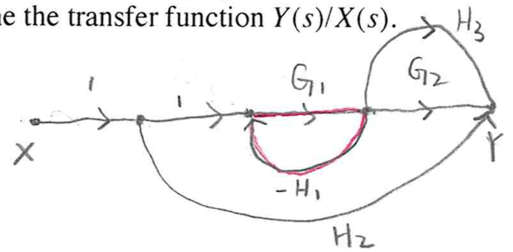
16.413 Problem Set #3

1. Consider the block diagram below.



(a) Evaluate the signal flow graph
(b) Determine the transfer function $Y(s)/X(s)$.

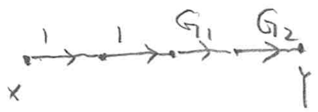
Solution : (a)



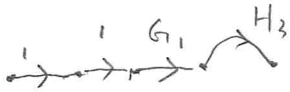
(b) Loop :

$L_1 = G_1(-H_1)$ $\quad \Delta = 1 - L_1 = 1 + G_1H_1$

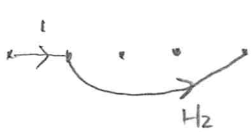
$\frac{Y(s)}{X(s)}$: Forward path :



$P_1 = G_1G_2 \quad \Delta_1 = 1$



$P_2 = G_1H_3 \quad \Delta_2 = 1$



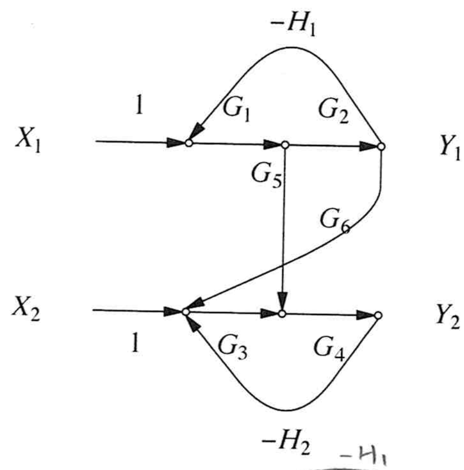
$P_3 = H_2 \quad \Delta_3 = 1 - L_1 = 1 + G_1H_1$

$$\frac{Y(s)}{X(s)} = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3}{\Delta} = \frac{1G_1G_2 + G_1H_3 + 1 + G_1H_1}{1 + G_1H_1}$$

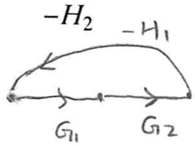
2. Given the signal flow graph below determine the transfer matrix A where $A_{ij} = Y_i/X_j$.

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

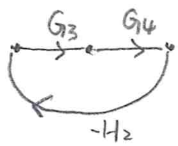
Note that $A_{ij} = Y_i/X_j$ given that all other inputs are equal to zero.



Solution: Loop:



$L_1 = G_1 G_2 (-H_1)$



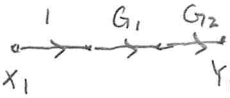
$L_2 = G_3 G_4 (-H_2)$

$\Delta = 1 - (L_1 + L_2) + L_1 L_2$

$A_{11} = \left(\frac{Y_1}{X_1}\right)\Big|_{X_2=0}$

$= \frac{P_1 \Delta_1}{\Delta}$

Forward path:



$P_1 = G_1 G_2$

$\Delta_1 = 1 - L_2 = 1 + G_3 G_4 H_2$

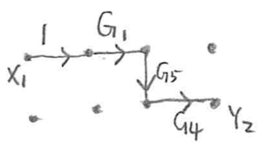
$A_{12} = \left(\frac{Y_1}{X_2}\right)\Big|_{X_1=0} = 0$

Forward path: $P_1 = 0$

$A_{21} = \left(\frac{Y_2}{X_1}\right)\Big|_{X_2=0}$

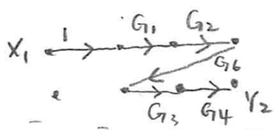
$= \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$

Forward path:



$P_1 = G_1 G_5 G_4$

$\Delta_1 = 1$

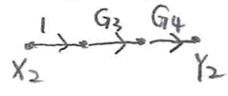


$P_2 = G_1 G_2 G_6 G_3 G_4$

$\Delta_2 = 1$

$A_{22} = \left(\frac{Y_2}{X_2}\right)\Big|_{X_1=0} = \frac{P_1 \Delta_1}{\Delta}$

Forward path:



$P_1 = G_3 G_4$

$\Delta_1 = 1 - L_1 = 1 + G_1 G_2 H_1$

3. Given the system equations

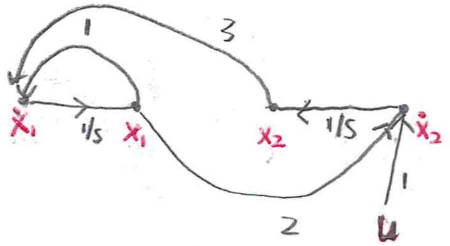
$$\frac{dx_1}{dt} = x_1 + 3x_2$$
$$\frac{dx_2}{dt} = 2x_1 + u$$

a. Draw a signal-flow graph representation of the system where $U(s)$ is the input and $X_1(s)$ is the output. You may assume zero initial conditions.

b. Find the transfer function $X_1(s)/U(s)$ using Mason's Gain formula. Check your result using an algebraic approach.

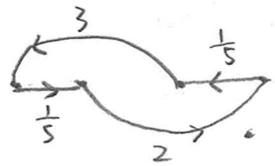
Solution : a. $\dot{X}_1 = X_1 + 3X_2$

$\dot{X}_2 = 2X_1 + u$



Loop:

$L_1 = \frac{1}{s}$



$L_2 = \frac{6}{s^2}$

$\Delta = 1-(L_1+L_2) = 1-(\frac{1}{s}+\frac{6}{s^2})$

$\frac{X_1(s)}{U(s)}$: Forward path:

$P_1 = \frac{1}{s}\cdot 3\cdot\frac{1}{s} = \frac{3}{s^2}$

$\Delta_1 = 1$

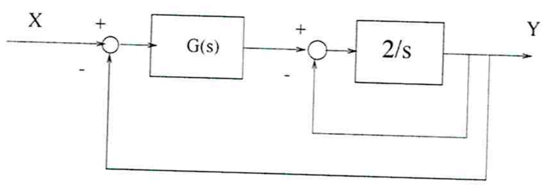
$\frac{X_1}{U} = \frac{P_1\Delta_1}{\Delta} = \boxed{\frac{3}{s^2-s-6}}$

Algebraic approach:

$\dot{X}_1 = X_1 + 3X_2$
$\dot{X}_2 = 2X_1 + u$
$\Rightarrow$ $sX_1 = X_1 + 3X_2$
$sX_2 = 2X_1 + U$
$\Rightarrow$ $(s-1)X_1 - 3X_2 = 0$
$-2X_1 + sX_2 = U$

$$\begin{bmatrix} s-1 & -3 \\ -2 & s \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ U \end{bmatrix} \Rightarrow X_1 = \frac{\begin{vmatrix} 0 & -3 \\ U & s \end{vmatrix}}{\begin{vmatrix} s-1 & -3 \\ -2 & s \end{vmatrix}} = \frac{3U}{s^2-s-6} \Rightarrow \boxed{\frac{X_1}{U} = \frac{3}{s^2-s-6}}$$

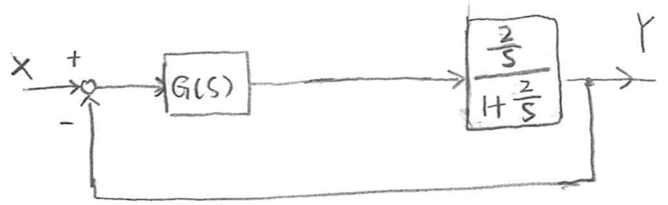
4. For the system below



a. Determine the transfer function Y/X

b. If the error is defined as $e(t) = x(t) - y(t)$ determine a $G(s)$ such that $e(\infty) = 1/2$ when $x(t) = t^2 u(t)$.

Solution : a.



$$\frac{Y}{X} = \frac{G(s)\frac{2}{s+2}}{1+G(s)\frac{2}{s+2}}$$

b. $E = X - Y \quad \frac{E}{X} = 1 - \frac{Y}{X} = 1 - \frac{G(s)\frac{2}{s+2}}{1+G(s)\frac{2}{s+2}} = \frac{1}{1+G(s)\frac{2}{s+2}}$

$x(t) = t^2 u(t)$

$$X(s) = \frac{2}{s^3}$$

$$e(\infty) = \lim_{s \to 0} sE = \lim_{s \to 0} \frac{2}{s^2} \cdot \frac{1}{1+G\frac{2}{s+2}} = \lim_{s \to 0} \frac{2}{s^2 + s^2 G \frac{2}{s+2}} = \frac{1}{2}$$

so $G(s) = \frac{K}{s^2}$

$\frac{2}{K} = \frac{1}{2} \Rightarrow \boxed{K = 4}$ $\qquad \boxed{G(s) = \frac{4}{s^2}}$