1.(a) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $P = 1$

(b) $R = \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix}$

$$Q - RBP^{-1}B^TR + RA + A^TR = 0$$

$$0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 \begin{bmatrix} 0 & 1 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} + \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} r_{12}^2 & r_{12}r_{22} \\ r_{12}r_{22} & r_{22}^2 \end{bmatrix} + \begin{bmatrix} 0 & r_{11} \\ r_{11} & 2r_{12} \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 - r_{12}^2 & r_{11} - r_{12}r_{22} \\ r_{11} - r_{12}r_{22} & 2r_{12} - r_{22}^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow R = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}$$

$$[k] = RBP^{-1} = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$$

$$u = -k^T \underline{x}$$

$$u = -\begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -x_1 - \sqrt{2}\, x_2$$

2

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix}$$

$$Q - RBP^{-1}B^TR + RA + A^TR = 0$$

$$0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} + \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix}$$

$$0 = -r_{11}[r_{12} + 2r_{11}] - r_{12}[r_{12} + r_{11}] + 2r_{12} + 1$$

$$-r_{12}[r_{22} + 2r_{12}] - r_{22}[r_{22} + r_{12}] + 2r_{22} + 2r_{12}$$

$$\begin{bmatrix} -r_{12}^2 + (2 - 2r_{11})r_{12} - 2r_{11}^2 + 1 & \text{Same} \\ (-r_{12} - r_{11} + 1)r_{22} - r_{12}^2 + (1 - 2r_{11})r_{12} + r_{11} & -r_{22}^2 + (2 - 2r_{12})r_{22} - 2r_{12}^2 + 2r_{12} \end{bmatrix}$$

3.

(a) $H = x_1^2 + u^2 + \lambda_1 x_2 + \lambda_2(-2x_1 - 3x_2 + u)$

(b) $\frac{\partial H}{\partial u} = 0 = 2u + \lambda_2 = 0$

$$u^o = -\frac{\lambda_2}{2}$$

(c) $H^o = x_1^2 + \frac{\lambda_2^2}{4} + \lambda_1 x_2 + \lambda_2\left[-2x_1 - 3x_2 - \frac{\lambda_2}{2}\right]$

(d) $\dot{x}_1 = x_2$

$\dot{x}_2 = -2x_1 - 3x_2 - \frac{\lambda_2}{2}$

$\dot{\lambda}_1 = -\left[x_1 2 \quad -2\lambda_2\right] \Leftarrow -\dot{\lambda}_1 = \frac{\partial H}{\partial x_1}$

$\dot{\lambda}_2 = -\left[\lambda_1 \quad -3\lambda_2\right] \Leftarrow -\dot{\lambda}_2 = \frac{\partial H}{\partial x_2}$

$$\underline{q} := \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} \qquad \underline{\dot{q}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix}$$

$$\underline{\dot{q}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & -\frac{1}{2} \\ -2 & 0 & 0 & +2 \\ 0 & 0 & -1 & +3 \end{bmatrix} \underline{q}$$

$$\underline{q}(0) = \begin{bmatrix} 0 \\ 0 \\ \lambda_1(0) \\ \lambda_2(0) \end{bmatrix} \qquad \underline{q}(1) = \begin{bmatrix} 1 \\ 0 \\ \lambda_1(1) \\ \lambda_2(1) \end{bmatrix}$$

$$\begin{array}{ccccc} \lambda & -1.9 & 1.18 & 1.9 & -1.18 \end{array}$$

$$E = \begin{bmatrix} -.43 & -0.04 & -0.13 & -2.74 \\ +.82 & -0.04 & -0.24 & 3.22 \\ -.37 & 0.95 & 3.14 & -3.31 \\ -.08 & 0.58 & 2.86 & -0.79 \end{bmatrix}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{-1.9t} & & & \\ & e^{1.18t} & & \\ & & e^{1.9t} & \\ & & & e^{-1.18t} \end{bmatrix}$$

$$\Phi = E^{-1} e^{\Lambda t} E$$

$$\Phi(1) = E^{-1} \cancel{\Phi(1)}\, e^{\Lambda t}\Big|_{t=1} E = \begin{bmatrix} 7.64 & 0.45 & 0.63 & 27.78 \\ -8.0 & 15.71 & 51.24 & -61.79 \\ 1.33 & -2.83 & -9.08 & 10.80 \\ -1.13 & -0.15 & -0.38 & -3.87 \end{bmatrix}$$

$$\begin{bmatrix} x_1(1) \\ x_2(1) \\ \hline \lambda_1(1) \\ \lambda_2(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \hline \lambda_1(1) \\ \lambda_2(1) \end{bmatrix} = \begin{bmatrix} \Phi(1) \end{bmatrix} \begin{bmatrix} \cancel{x_1(0)}^{\,0} \\ \cancel{x_2(0)}^{\,0} \\ \hline \lambda_1(0) \\ \lambda_2(0) \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cancel{7.64\, x_1(0)} + \cancel{0.45\, x_2(0)} + 0.63\,\lambda_1(0) + 27.78\,\lambda_2(0) \\ \cancel{-8.0\, x_1(0)} + \cancel{15.71\, x_2(0)} + 51.24\,\lambda_1(0) - 61.79\,\lambda_2(0) \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.63 & 27.78 \\ 51.24 & -61.79 \end{bmatrix} \begin{bmatrix} \lambda_1(0) \\ \lambda_2(0) \end{bmatrix}$$

$$\lambda_1(0) = 4.2 \times 10^{-2}$$

$$\lambda_2(0) = 3.5 \times 10^{-2}$$