University of Massachusetts Lowell
Department of Electrical and Computer Engineering

16.413 Linear Feedback

Problem set 6

1. Given the state equations

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -14 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \alpha_1 & \alpha_2 \\ 0 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\underline{y} = \begin{bmatrix} \beta & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{bmatrix} \underline{x}$$

   a. Determine the eigenvalues and eigenvectors of the system matrix.

   b. Determine the values for α_1 and α_2 for the system to be controllable.

   c. Determine the values for β the system to be observable.

2. Given the state equations

$$\frac{d\underline{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y = [1 \;\; 0]\underline{x}$$

   a. Determine the eigenvectors of the system matrix.

   b. Determine STM.

   c. If the initial condition are equal to zero and $u(t) = \delta(t)$. Determine $y(t)$.

3. Consider the differential equation

$$\frac{d^4 z}{dt^4} + 3\frac{d^3 z}{dt^3} + 6\frac{d^2 z}{dt^2} + 2\frac{dz}{dt} + z = 6u$$

The ouput y is given by the expression

$$y = \frac{d^3 z}{dt^3} + 6z$$

Represent the ODE in phase-variable form such that

$$\frac{d\underline{x}}{dt} = A\underline{x} + B\underline{u}$$
$$y = C\underline{x}$$

a. Define $\underline{x}$

b. Define $\underline{u}$

c. Find the matrix A, C and the matrix B

PS #6

① $\dot{\underline{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -14 & -7 \end{bmatrix}}_{A} \underline{x} + \underbrace{\begin{bmatrix} \alpha_1 & \alpha_2 \\ 0 & -2 \\ -2 & -1 \end{bmatrix}}_{B} \underline{u} \; ; \; \underline{y} = \underbrace{\begin{bmatrix} \beta & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{bmatrix}}_{C} \underline{x}$

(a) $\lambda_1 = -2 \quad \underline{e}_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ $\quad \lambda_2 = -1 \quad \underline{e}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $\quad \lambda_3 = -4 \quad \underline{e}_3 = \begin{bmatrix} 1 \\ -4 \\ 16 \end{bmatrix}$

(b) $E = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -1 & -4 \\ 4 & 1 & 16 \end{bmatrix} \; ; \; E^{-1} = \begin{bmatrix} -2 & -5/2 & -1/2 \\ 8/3 & 2 & 1/3 \\ 1/3 & 1/2 & 1/6 \end{bmatrix}$

377

$$E^{-1}B = \begin{bmatrix} -2 & -5/2 & -1/2 \\ 8/3 & 2 & 1/3 \\ 1/3 & 1/2 & 1/6 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \\ 0 & -2 \\ -2 & -1 \end{bmatrix}$$

controllable if

$$= \begin{bmatrix} -\alpha_1 2 + 1 & -2\alpha_2 + 10 + 1/2 \\ \alpha_1 8/3 - 2/3 & \alpha_2 \frac{8}{3} - 4 + 1/3 \\ \alpha_1/3 - 1/3 & \alpha_2/3 - 1 - 1/6 \end{bmatrix} \begin{matrix} \Rightarrow \alpha_1 \neq \frac{1}{2} \; \& \; \alpha_2 \neq \frac{11}{4} \\ \Rightarrow \alpha_1 \neq \frac{1}{4} \; \& \; \alpha_2 \neq \frac{13}{8} \\ \Rightarrow \alpha_1 \neq 1 \; \& \; \alpha_2 \neq 7/2 \end{matrix}$$

(c) $$CE = \begin{bmatrix} \beta & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -2 & -1 & -4 \\ 4 & 1 & 16 \end{bmatrix} = \begin{bmatrix} \beta + 14 & \beta + 3 & \beta + 60 \\ 4 & 1 & 16 \\ -10 & -4 & -28 \end{bmatrix}$$

$$\begin{matrix} \Downarrow & \Downarrow & \Downarrow \\ \text{non zero} & \text{non zero} & \text{non zero} \end{matrix}$$

<u>obs for all β</u>

2

(a) $\Lambda = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad \lambda_1 = -5 \quad \underline{e}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$
$\quad \lambda_2 = -1 \quad \underline{e}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(b) $\underline{\Phi} = E e^{\Lambda t} E^{-1}$

$$\underline{\Phi} = \begin{bmatrix} 1 & 1 \\ -5 & -1 \end{bmatrix} \begin{bmatrix} e^{-5t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -5 & -1 \end{bmatrix}^{-1}$$

$$\underline{\Phi} = \begin{bmatrix} \frac{1}{4}(5e^{-t} - e^{-5t}) & \frac{1}{4}(e^{-t} - e^{-5t}) \\ \frac{1}{4}(5e^{-5t} - 5e^{-t}) & \frac{1}{4}(5e^{-5t} - e^{-t}) \end{bmatrix}$$

(c) $\underline{x}(t) = \cancel{\underline{\Phi}\,\underline{x}(0)}^{\,0} + \underline{\Phi} * \begin{bmatrix} 1 \\ 0 \end{bmatrix} \delta(t)$

$$= \begin{bmatrix} \frac{1}{4}(5e^{-t} - e^{-5t}) \\ \frac{1}{4}(5e^{-5t} - 5e^{-t}) \end{bmatrix} * \delta(t)$$

$$\underline{x}(t) = \frac{1}{4} \begin{bmatrix} 5e^{-t} - e^{-5t} \\ 5e^{-5t} - 5e^{-t} \end{bmatrix}$$

3

$z = x_1$
$z' = x_2$
$z'' = x_3$
$z''' = x_4$

$$\dot{\underline{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -6 & -3 \end{bmatrix}}_{A} \underline{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \end{bmatrix}}_{B} u$$

y

$$y = \underbrace{\begin{bmatrix} 6 & 0 & 0 & 1 \end{bmatrix}}_{C} \underline{x}$$

(a) $\underline{x} = \begin{bmatrix} z \\ z' \\ z'' \\ z''' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

(b) $\underline{u} = u$

(c)