University of Massachusetts Lowell
Department of Electrical and Computer Engineering

16.413 Linear Feedback

Problem set 5

1. Determine the root loci for the closed-loop unity negative feedback system $H(s) = 1$

$$G(s) = \frac{K}{s(s+1)(s^2+4s+5)}$$

2. Determine the root loci for the closed-loop unity negative feedback system $H(s) = 1$

$$G(s) = \frac{K(s+9)}{s(s^2+4s+11)}$$

   a. By adjustment of the gain K locate the closed-loop poles on the root loci such that the dominant closed-loop poles have a damping factor equal to 0.5.

3. Consider the closed-loop unity negative feedback system where

$$G(s) = \frac{1}{s^2(s+4)}$$

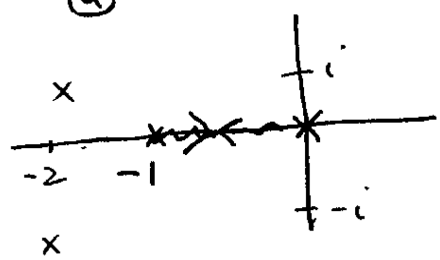
Design a compensator $H(s)$ such that the unit-step response of the system has an overshoot $\leq$ 25% and a setting time of 5 seconds or less.

(5)

(1) $GH = \dfrac{k}{s(s+1)(s^2+4s+5)}$ (a)

real-ln
re

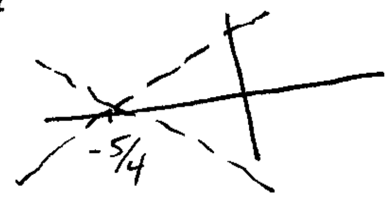
poles: $s = \begin{cases} 0 \\ -1 \\ -2 \pm i \end{cases}$



(b) asymp.

$\sigma = \dfrac{0 - 1 - 2 + i - 2 - i}{4 - 0} = \dfrac{-5}{4}$

$\measuredangle = \dfrac{\pm(2n+1)\pi}{4-0} = \begin{cases} \pm \pi/4 \\ \pm 3\pi/4 \end{cases}$



(c) jw-axis crossing

ch, eqn $GH + 1 = 0$

$s^4 + 5s^3 + 9s^2 + s5 + k = 0$

s^4	1	9	k
s^3	5	5	
s^2	$\cancel{\frac{45-5}{5}}$ 8	k	
s^1	$\frac{40-5k}{8}$		
s^0	k		

$\Rightarrow P(s) = 8s^2 + k = 0 \Rightarrow s^2 + 1 = 0$ $\boxed{s = \pm j}$

$\Rightarrow \dfrac{40-5k}{8} = 0 \Rightarrow \boxed{k = 8}$

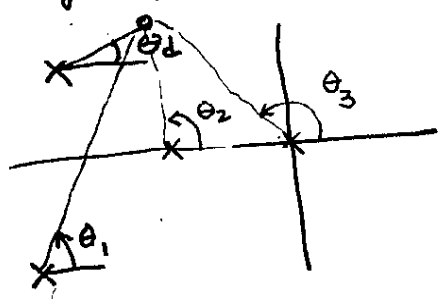
(d) Breakaway pt

$GH + 1 = 0 \Rightarrow k = -\{s^4 + 5s^3 + 9s^2 + 5s\}$

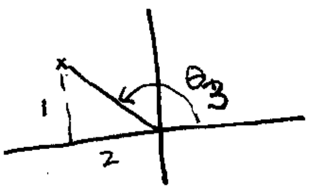
$\dfrac{dk}{ds} = 0 = 4s^3 + 15s^2 + 18s + 5 = 0 \Rightarrow \boxed{s = -0.39}$

$k = -\{s^4 + 5s^3 + 9s^2 + 5s\}\Big|_{s=-0.39} = \boxed{0.85}$

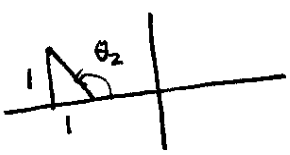
(e) angle of departure



$$\pm(2n+1)\pi = -\theta_d - \theta_1 - \theta_2 - \theta_3$$



$$\theta_3 = \pi - \tan^{-1}\left(\frac{1}{2}\right)$$

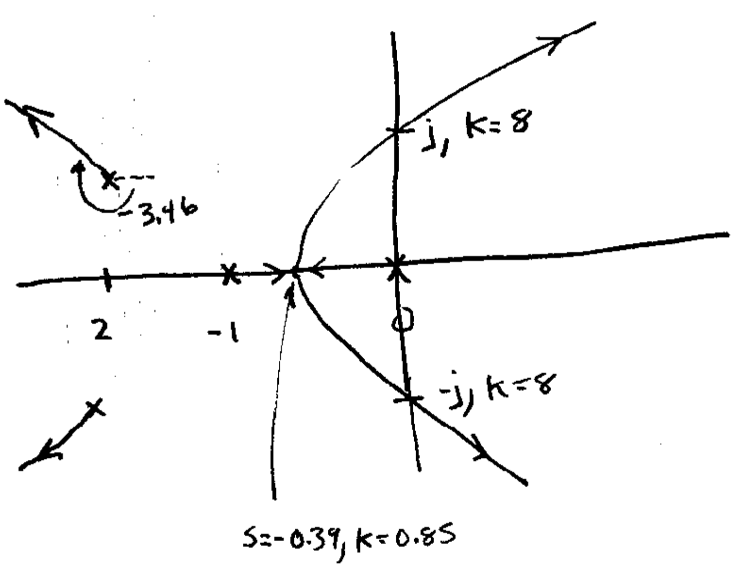


$$\theta_2 = \pi - \tan^{-1}\left(\frac{1}{1}\right)$$

$$\theta_1 = \frac{\pi}{2}$$

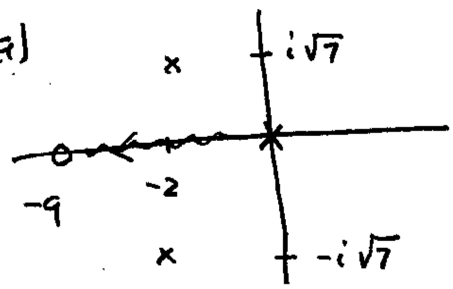
$$\pm(2n+1)\pi = -\theta_d - \pi + \tan^{-1}\left(\frac{1}{2}\right) - \pi + \tan^{-1}(1) - \frac{\pi}{2}$$

$$\theta_d = -3.46$$



2

(a)



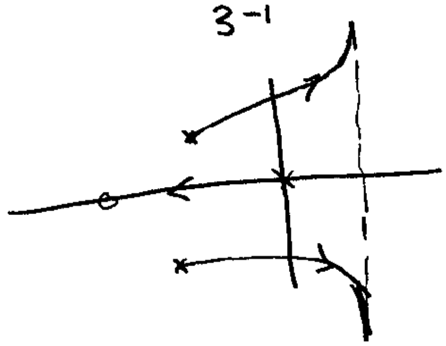
zeros: $s = -9$

poles: $s = 0$

$s = -2 \pm i\sqrt{7}$

(b) asymp $\sigma = \dfrac{+9-0-2-i-2+i}{3-1} = \dfrac{5}{2}$

$\phi = \dfrac{\pm(2n+1)\pi}{3-1} = \pm\dfrac{\pi}{2}$

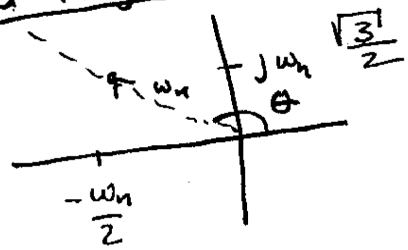


note we have complex conjugate pair

given $\zeta = \frac{1}{2}$

$$s = -\frac{\omega_n}{2} \pm j\omega_n\sqrt{1-\frac{1}{4}} = -\frac{\omega_n}{2} \pm j\omega_n\frac{\sqrt{3}}{2}$$

pole movement target

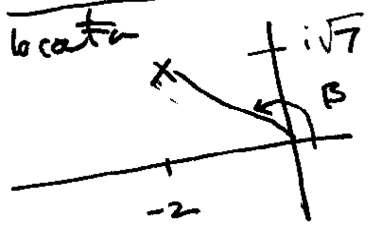


$$\theta = \pi - \tan^{-1}\left(\sqrt{3}\right)$$

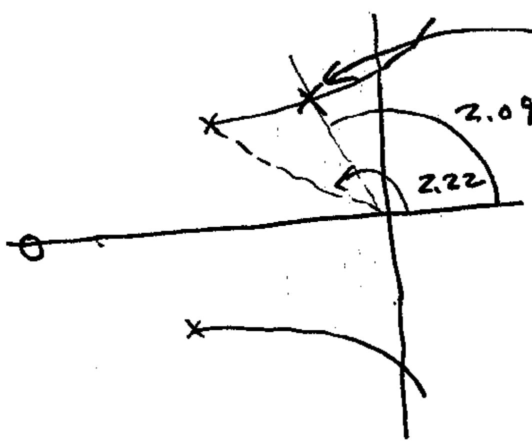
$$\theta = \pi - \frac{\pi}{3}$$

$$\boxed{\theta = \frac{2\pi}{3}} = 2.09$$

original pole location



$$\beta = \pi - \tan^{-1}\left(\frac{\sqrt{7}}{2}\right) = 2.218$$



possible intersection

note $\theta < \beta$ to intersect

$$\boxed{s = \frac{\omega_n}{2}(-1 - j\sqrt{3})}$$

Characteristic eqn

$$s^3 + 4s^2 + (K+11)s + 9K \Big|_{s = \frac{\omega_n}{2}(-1-j\sqrt{3})} = 0$$

<u>real part</u>

$$(\omega_n - 18)K - 2\omega_n^3 + 4\omega_n^2 + 11\omega_n = 0$$

<u>imag part</u>

$$\frac{\sqrt{3}}{2}\omega_n\left[K - 4\omega_n + 11\right] = 0 \Rightarrow \boxed{K - 4\omega_n + 11 = 0}$$

$$\therefore \boxed{K = 4\omega_n - 11}$$

real part $K = 4\omega_n - 11$

$$(\omega_n - 18)(4\omega_n - 11) - 2\omega_n^3 + 4\omega_n^2 + 11\omega_n = 0$$

$$\Downarrow$$

$$\omega_n^3 - 4\omega_n^2 + 36\omega_n - 99 = 0$$

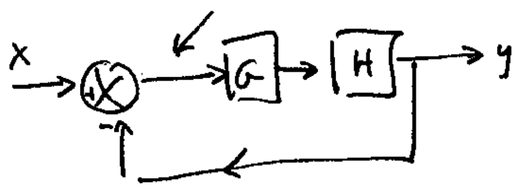
$$\Downarrow$$

$$\boxed{\omega_n = 3}$$

$$\Downarrow$$

$$\boxed{K = 4.3 - 11 = 1}$$

3.



$$\frac{Y}{X} = \frac{GH}{1+GH}$$

$$Y = \frac{1}{s}\,\frac{GH}{1+GH}$$

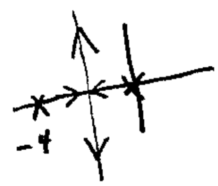
$$y(\infty) = \lim_{s\to 0} \frac{GH}{1+GH} = \lim_{s\to 0} \frac{H\frac{1}{s^2(s+4)}}{1+\frac{1}{s^2(s+4)}H} \Rightarrow 1 \text{ for } H = \begin{cases} k \\ ks \Leftarrow \end{cases}$$

(a) $H = k$

$$GH = \frac{k}{s^2(s+4)}$$

(b) $H = ks$

$$GH = \frac{k}{s(s+4)}$$



$$\frac{Y}{X} = \frac{k}{s^2+4s+k} \qquad \omega_n = \sqrt{k}, \quad 2\zeta\omega_n = 4 \Rightarrow \boxed{\zeta = \frac{2}{\sqrt{k}}}$$

$$M_p = e^{-\frac{\zeta\omega_n\pi}{\omega_n\sqrt{1-\zeta^2}}} \le 0.25$$

$$\boxed{e^{-\frac{2\pi}{\sqrt{k}\sqrt{1-4/k}}} \le 0.25} \Rightarrow \frac{-2\pi}{\sqrt{k-4}} \le \ln(0.25)$$

$$t_s = \frac{-\ln(\epsilon)}{\zeta\omega_n}$$

$$\boxed{\frac{-\ln(\epsilon)}{2} < 5}$$

say $\epsilon = 0.02$

$\frac{3.9}{2} < 5$ ✓

∴ satisfied @ 2%