University of Massachusetts Lowell
Department of Electrical and Computer Engineering

16.413 Linear Feedback

Problem set 4

1. The open-loop transfer function for three unity feedback control systems is given below. Evaluate and compare the unit step, impulse and ramp responses. Which system yields the fastest rise-time with the smallest overshoot for the step input. Using a s-plane analysis of the closed-loop transfer function justify your answer.

a. $G(s) = \dfrac{5}{s(5s+1)}$

b. $G(s) = \dfrac{5(1+0.8s)}{2(5s+1)}$

c. $G(s) = \dfrac{1}{s(s+1)}$

2. Given the ODE

$$\frac{d^2y}{dt^2} + 2\frac{dy(t)}{dt} + 3y = 0$$

where $y(0) = 2$ and $dy(0)/dt = 1$ determine the solution $y(t)$.

3. Determine the conditions on the gain K for the unity feedback system which will render the closed-loop system stable. The open-loop transfer function is

$$G(s) = \frac{K}{s(s+1)(2s+3)}$$

4. Given the characteristic equation for the closed-loop transfer function determine the conditions on the the gain K for the system to be stable.

a. $s^4 + 22s^3 + 10s^2 + 2s + K = 0$

b. $s^4 + 20Ks^3 + 5s^2 + (10+K)s + 15 = 0$

c. $s^3 + (K+0.5)s^2 + 5Ks^2 + +50 = 0$

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1(a) $\bar{Y} = \bar{X}\left[\dfrac{1}{s^2+\frac{s}{5}+1}\right]$

$\omega_n = 1$

$2\xi\omega_n = \frac{1}{5} \Rightarrow \xi = \frac{1}{10}$

$\omega_d = \omega_n\sqrt{1-\xi^2}$

$\omega_d = 0.99$

$$t_r = \frac{\pi - \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{\omega_d} = \frac{\pi - \tan^{-1}(10(.99))}{.99} = 1.68$$

$$M_p = e^{\frac{-\xi\omega_n}{\omega_n\sqrt{1-\xi^2}}\pi} = e^{-\frac{\pi}{10}} = \cancel{1.37}\,0.73$$

$x(t) = \delta(t)$

$$y(t) = \frac{10}{3\sqrt{11}} e^{-t/10} \sin\left[\frac{3\sqrt{11}}{10}t\right] u(t)$$

$x(t) = u(t)$

$$y(t) = 1 - e^{-t/10}\left[\cos\left[\frac{3\sqrt{11}}{10}t\right] + \frac{\sin\left(\frac{3\sqrt{11}}{10}t\right)}{3\sqrt{11}}\right] u(t)$$

$x(t) = t\,u(t)$

$$y(t) = \left\{t - \frac{1}{5} + \frac{e^{-t/10}}{5}\left[\cos\left[\frac{3\sqrt{11}}{10}t\right] - \frac{49}{3\sqrt{11}}\sin\left[\frac{3\sqrt{11}}{10}t\right]\right]\right\} u(t)$$

(b) $\bar{Y} = \bar{X}\left[\dfrac{4s+5}{14s+7}\right] = \bar{X}\left[1 + \dfrac{3/4}{s+1/2}\right]\dfrac{4}{14}$

<u>no rise and overshoot</u>

(i) $x(t) = \delta(t)$

$$y(t) = \frac{4}{14}\delta(t) + \frac{3}{4}\left(\frac{4}{14}\right)e^{-t/2}\,u(t)$$

(ii) $x(t) = u(t)$

$$y(t) = \frac{4}{14}\left[u(t) + \frac{3}{2}\left\{1 - e^{-t/2}\right\}u(t)\right]$$

(iii) $x(t) = t\,u(t)$

$$y(t) = \frac{4}{14}\left[t\,u(t) + \frac{3}{4}\left[e^{-t/2} + \frac{1}{2}t - 1\right]u(t)\right]$$

1c $\bar{Y} = \bar{X}\left[\dfrac{1}{s^2+s+1}\right]$ $\quad \omega_n = 1$, $\quad 2\zeta\omega_n = 1 \Rightarrow \zeta = \frac{1}{2}$ $\quad \Big| \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$, $\quad \omega_d = \sqrt{\frac{3}{4}}$

$$t_r = \frac{\pi - \tan^{-1}\left(\frac{\sqrt{\frac{3}{4}}}{1/2}\right)}{\sqrt{3/4}} = 2.41$$

$$M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} = e^{-\frac{\pi}{2}\frac{1}{\sqrt{3/4}}} = 1.6 \times 10^{-1}$$

$x(t) = \delta(t)$

$$y(t) = \frac{2}{\sqrt{3}} e^{-t/2} \sin\left[\sqrt{\frac{3}{4}}\, t\right] u(t)$$

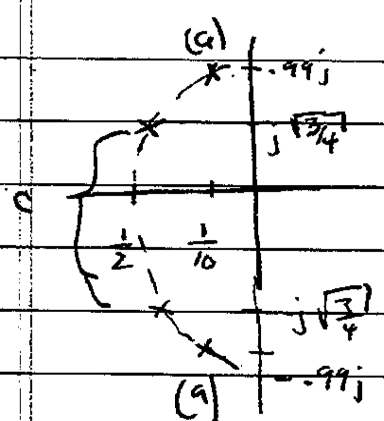
$x(t) = u(t)$

$$y(t) = \left\{1 - e^{-t/2}\left[\cos\left(\sqrt{\frac{3}{4}}\, t\right) + \frac{\sin\left(\sqrt{\frac{3}{4}}\, t\right)}{\sqrt{3}}\right]\right\} u(t)$$

$x = t\,u(t)$

$$y(t) = \left\{t - 1 + e^{-t/2}\left[\cos\left(\sqrt{\frac{3}{4}}\, t\right) - \frac{\sin\left(\sqrt{\frac{3}{4}}\, t\right)}{\sqrt{3}}\right]\right\} u(t)$$

	t_r	M_p	
a	1.68	0.73	⇐ fastest rise time
b	na	na	
c	2.41	0.16	⇐ smallest overshoot



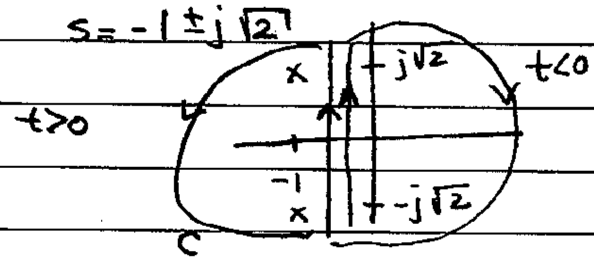
2.

$$y'' + 2y' + 3y = 0 \overset{\mathcal{L}}{\Rightarrow} s^2 \bar{Y} - s\,y(0) - y'(0) + 2(s\bar{Y} - y(0)) + \bar{Y}3 = 0$$

$$[s^2 + 2s + 3]\bar{Y} = s\,y(0) + y'(0) + 2y(0)$$

$$\bar{Y} = \frac{(s+2)\,y(0) + y'(0)}{s^2 + 2s + 3} = \frac{(s+2)2 + 1}{s^2 + 2s + 3}$$

poles: $s = -1 \pm j\sqrt{2}$



$\boxed{y = 0 \quad t < 0}$

$\underline{\underline{t > 0}}$

$$y(t) = \frac{1}{2\pi i} \oint_C \bar{Y}(s) e^{st} ds = \bar{Y} e^{st}(s+1+j\sqrt{2})\Big|_{s=-1-j\sqrt{2}} + \bar{Y} e^{st}(s+1-j\sqrt{2})\Big|_{s=-1+j\sqrt{2}}$$

$$\boxed{y(t) = e^{-t}\left[2\cos(\sqrt{2}t) + \frac{3}{\sqrt{2}}\sin(\sqrt{2}t)\right]u(t)}$$

3. $Y = \frac{G}{1+G} \Rightarrow 1+G=0 \Rightarrow s(s+1)(2s+3)+k=0$

$2s^3+5s^2+3s+k=0$

s^3	2	3
s^2	5	k
s^1	$\frac{5\cdot 3-2\cdot k}{5}$	
s^0	k	

s^1: $\Rightarrow 15-2k>0 \Rightarrow \frac{15}{2}>k$

s^0: $\Rightarrow k>0$

$k>0 \cap \frac{15}{2}>k \Rightarrow \boxed{\frac{15}{2}>k>0}$

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(a)

s^4	1	10	k
s^3	22	2	
s^2	$\frac{22\cdot 10-1\cdot 2}{\cancel{22}} \rightarrow \frac{218}{22}$	$\frac{\cancel{22}\cdot k}{\cancel{22}} \rightarrow k$	
s^1	$\frac{\frac{218}{22}\cdot 2-22\cdot k}{\frac{218}{22}}$		$2\cdot\frac{218}{22^2} > k$
s^0	k		$k>0$

$$\boxed{\frac{(2)(218)}{22^2} > k > 0}$$

4(b)

$$
\begin{array}{c|ccc}
s^4 & 1 & 5 & 15 \\
s^3 & 20k & 10+k & \\
s^2 & \dfrac{100k-(10+k)}{20k} \rightarrow \dfrac{99k-10}{20k} & 15 & \\
s^1 & \dfrac{\left(\dfrac{99k-10}{20k}\right)(10+k)-300k}{\dfrac{99k-10}{20k}} & & \\
s^0 & 15 & &
\end{array}
$$

$20k > 0 \quad \Rightarrow \quad k > 0$

$99k-10 > 0 \quad \Rightarrow \quad k > \frac{10}{99}$

$-5901k^2 + 980k - 100 > 0$

$\Downarrow$

$$\left[k^2 - \frac{980}{5901}k + \frac{100}{5901}\right] < 0$$

$$\left(k - \frac{980}{2(5901)}\right)^2 - \left(\frac{980}{2(5901)}\right)^2 + \frac{100}{5901} < 0$$

$$\left(k - \frac{980}{2(5901)}\right)^2 < -\frac{50000}{4974543}$$

cannot be satisfied for any real value of k

Unstable for all k

4 (a) $s^3 + (6k + 0.5)s^2 + 50 = 0$

$$\begin{array}{c|cc}
s^3 & 1 & 0 \\
s^2 & (6k + \frac{1}{2}) & 50 \\
s^1 & \frac{-50}{(6k+\frac{1}{2})} & \\
s^0 & 50 &
\end{array}$$

$\Rightarrow 6k + \frac{1}{2} > 0$

$\Rightarrow \frac{-50}{6k+\frac{1}{2}} > 0$

$\left.\begin{array}{l} k > -\frac{1}{12} \\ -50 \not> 0 \end{array}\right\}$ unstable for all k

ie $6k + \frac{1}{2} > 0$ then $\frac{-50}{6k+1/2} < 0$ | $6k + \frac{1}{2} < 0$, $\frac{-50}{6k+\frac{1}{2}} > 0$